Exercises Lambda Calculus (week 7, 08.01.2014)

1. Define **true** $\triangleq \lambda xy.x (\equiv \mathsf{K})$ and **false** $\triangleq \lambda xy.y (= \mathsf{KI})$.

   (a) Given λ-terms P, Q, construct a λ-term $F_{P,Q}$ such that

   $$\begin{aligned} F_{P,Q}\mathbf{true} &= P; \\ F_{P,Q}\mathbf{false} &= Q. \end{aligned}$$

   (b) Construct a λ-term F_{neg} such that

   $$F_{\text{neg}}\mathbf{true} = \mathbf{false}\ \&\ F_{\text{neg}}\mathbf{false} = \mathbf{true}.$$

   (c) Construct a term F_{and} such that

   $$\begin{aligned} F_{\text{and}}\mathbf{true}\ \mathbf{true} &= \mathbf{true}; \\ F_{\text{and}}\mathbf{true}\ \mathbf{false} &= \mathbf{false}; \\ F_{\text{and}}\mathbf{false}\ \mathbf{true} &= \mathbf{false}; \\ F_{\text{and}}\mathbf{false}\ \mathbf{false} &= \mathbf{false}. \end{aligned}$$

2. (a) Construct a λ-term G such that

   $$\begin{aligned} G\ulcorner x\urcorner &= \mathbf{true} \\ G\ulcorner PQ\urcorner &= \mathbf{false} \\ G\ulcorner \lambda x.P\urcorner &= \mathbf{false}. \end{aligned}$$

   (b) Construct a λ-term V such that

   $$\begin{aligned} V\ulcorner x\urcorner &= \mathbf{true} \\ V\ulcorner PQ\urcorner &= V\ulcorner P\urcorner \\ V\ulcorner \lambda x.P\urcorner &= \mathbf{false}. \end{aligned}$$

   (c) Construct a λ-term H such that

   $$\begin{aligned} H\ulcorner x\urcorner &= \mathbf{true} \\ H\ulcorner PQ\urcorner &= V\ulcorner P\urcorner \\ H\ulcorner \lambda x.P\urcorner &= H\ulcorner P\urcorner. \end{aligned}$$

   (d) Compute $H\ulcorner S\urcorner$, $H\ulcorner \mathsf{Y}\urcorner$, where $\mathsf{Y} \triangleq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

   (e) Note that $\mathsf{Y} = \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))$. Compute

   $$H\ulcorner \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))\urcorner.$$

3.* (a) Every λ-term M is either of the form $M \equiv \lambda x_1 \dots x_n.yQ_1 \dots Q_m$ (with *head variable* y) or $M \equiv \lambda x_1 \dots x_n.(\lambda y.P)Q_0Q_1 \dots Q_m$ (with *head redex* $(\lambda y.P)Q$). Show this. In the first case M is a *head normal form*.

   (b) Show that in the first respectively second case one has $H\ulcorner M\urcorner = \mathbf{true}$ and $H\ulcorner M\urcorner = \mathbf{false}$.