Böhm's Theorem, Church's Delta, Numeral Systems, and Ershov Morphisms

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Abstract

In this note we work with untyped lambda terms under $\beta(\eta)$-conversion and consider the possibility of extending Böhm's theorem to infinite RE sets. Of course, it is well known that Böhm's theorem will fail in general for such sets even if it holds for all finite subsets. It turns out that generalizing Böhm's theorem to infinite sets involves three other superficially unrelated notions; namely, Church's delta, numeral systems, and Ershov morphisms. Our principal result is that Böhm's theorem holds for an infinite RE set V closed under beta conversion iff V can be endowed with the structure of a numeral system with predecessor iff there is a Church delta (conditional) for V iff every Ershov morphism with domain V can be represented by a lambda term. Along the way we prove a version of the Myhill-Shepherdson theorem for Ershov morphisms, and an approximation theorem for beta-eta morphisms by lambda terms.

1. Introduction

1.1. Definition. (i) The set of untyped closed lambda terms is denoted by $\Lambda^{\varnothing}$. A *combinator* is an element of $\Lambda^{\varnothing}$.

(ii) We denote congruence under beta conversion by $=$.

(iii) We write $:=$ for "equal by definition".

(iv) We define the following combinators.

$$\begin{array}{rcll} \mathbf{c}_n & := & \lambda fx.f^n x, & \text{the } \textit{Church numerals}. \\ \mathsf{U}^n_k & := & \lambda x_1 \ldots x_n.x_k, & \text{for } 1 \le k \le n, \text{ the } \textit{projections}. \\ \Omega & := & (\lambda x.xx)(\lambda x.xx). & \end{array}$$

The classical theorem of Böhm implies the following.

1.2. Theorem (Böhm [1968]). *For all combinators M_1 and M_2 having a β-nf (normal form) the following are equivalent.*

(i) *For all combinators N_1, N_2 there exist combinators $\vec{P}$ such that*

$$M_1\vec{P} = N_1 \;\&\; M_2\vec{P} = N_2.$$

(ii) *For all combinators N_1, N_2 there exists a combinator F such that*

$$FM_1 = N_1 \;\&\; FM_2 = N_2.$$

(iii) *There exists a combinator* F *such that*

$$FM_1 = \lambda xy.x \ \&\ FM_2 = \lambda xy.y.$$

(iv) $M_1 = M_2$ *is inconsistent with* $\lambda\beta$.
(v) $M_1 \neq_{\beta\eta} M_2$.
(vi) M_1 *and* M_2 *have distinct* $\beta\eta$*-nfs (normal forms).*

The only non-trivial implication is (vi)⇒(i), the core of Böhm's theorem, follows from Barendregt [1984] Theorem 10.4.2 and the fact that β and $\beta\eta$ normalizability are equivalent, ibidem Corollary 15.1.5.

For finite sets $\mathcal{F} = \{M_1, \ldots, M_n\}$ of combinators one has the following generalizaton.

1.3. Theorem (Böhm, Dezani-Ciancaglini, Peretti and Ronchi [1979]). *For all combinators* $M_1, \ldots, M_n$ *having a* β*-nf the following are equivalent.*

(i) *For all combinators* $N_1, \ldots, N_n$ *there exist combinators* $\vec{P}$ *such that*

$$M_1\vec{P} = N_1 \ \&\ \ldots\ \&\ M_n\vec{P} = N_n.$$

(ii) *For all combinators* $N_1, \ldots, N_n$ *there exists a combinator* F *such that*

$$FM_1 = N_1 \ \&\ \ldots\ \&\ FM_n = N_n.$$

(iii) $\mathcal{F}$ *is* separable, *i.e. there exists a combinator* F *such that*

$$FM_1 = \mathsf{U}^n_1 \ \&\ \ldots\ \&\ FM_n = \mathsf{U}^n_n.$$

(iv) $M_p = M_q$ *is inconsistent with* $\lambda\beta$*, for* $1 \leq p, q \leq n$ *with* $p \neq q$.
(v) $M_p \neq_{\beta\eta} M_q$*, for* $1 \leq p, q \leq n$ *with* $p \neq q$.
(vi) *The* $M_1, \ldots, M_n$ *have pairwise distinct* $\beta\eta$*-nfs.*

Again, the only non-trivial implication is (vi) ⇒ (i) and is proved in Böhm, Dezani-Ciancaglini, Peretti and Ronchi [1979], see Barendregt [1984], Corollary 10.4.14.

Separability for infinite sets can be defined as the existence of a definable 1-1 map (modulo β-conversion) to the Church numerals.

For infinite sets $\mathcal{F}$ of combinators the property of having distinct $\beta\eta$-nfs does not necessarily imply separability. For example this is the case with

$$\mathcal{F} = \{\Omega\mathbf{c}_n \mid n \in \mathbb{N}\}.$$

One may think this is caused by the fact that $\mathcal{F}$ consists of combinators without a nf, but this is not the case. An example of a set $\mathcal{F}$ of combinators in nf, such that even each finite subset is separable but not the whole set, is the collection of projections

$$\mathcal{F} = \{\mathsf{U}^n_k \mid n \in \mathbb{N} \ \&\ 1 \leq k \leq n\}.$$

1.4. DEFINITION. (i) $\#M$ is the Gödel number of a combinator M. We write $\ulcorner M \urcorner$ for $\mathbf{c}_{\#M}$[1].

(ii) There is an inverse E, called *Kleene's enumerator*, such that $\mathsf{E}\ulcorner M \urcorner = M$, for all combinators M, see Barendregt [1984] Theorem 8.1.6.

(iii) For $m, n \in \mathbb{N}$ we write $m \sim n \iff \mathsf{E}\mathbf{c}_m = \mathsf{E}\mathbf{c}_n$.

(iv) A *partial Ershov morphisms* $\Phi : \Lambda^{\emptyset}/= \to \Lambda^{\emptyset}/=$ is a partial map such that for some partial recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ and all combinators M

$$\Phi(M) \cong \mathsf{E}(\mathbf{c}_{\varphi(\#M)}),$$

where $P \cong Q$ means that if one of the two expressions P, Q is defined, then so is the other and $P = Q$. This is implied by $\#\Phi(M) \simeq \varphi(\#M)$, with a similar meaning for $\simeq$. Note that Φ is completely determined by a φ such that

$$n \sim m \Rightarrow \varphi(n) \sim \varphi(m).$$

We write $\Phi(M)\downarrow, \varphi(m)\downarrow$ for convergence of the partial functions (being defined); similarly $\Phi(M)\uparrow, \varphi(m)\uparrow$ for divergence (being undefined).

In the present paper the following will be proved.

1.5. THEOREM. *Suppose that V is an RE set of combinators closed under β conversion. Then the following are equivalent.*

(i) *V forms an adequate numeral system, i.e. there are combinators $0, S, P, Z$ such that for all $k \in \mathbb{N}$*

$$\begin{array}{rcl} V &=& \{S^n O \mid n \in \mathbb{N}\}, \\ P(S^{k+1}O) &=& S^k O, \\ ZO &=& \mathsf{U}^2_1, \\ Z(S^{k+1}O) &=& \mathsf{U}^2_2. \end{array}$$

(ii) *For every morphism Φ with $dom(\Phi) \subseteq V$ there is an $F \in \Lambda^{\emptyset}$ such that*

$$\forall M \in V. \Phi(M) = FM.$$

(iii) *There exists a combinator δ such that*

$$\begin{array}{rcll} \delta M\ N &=& \mathsf{U}^2_1, & \textit{if } M = N; \\ &=& \mathsf{U}^2_2, & \textit{else.} \end{array}$$

(iv) *There is a morphism Φ with $dom(\Phi) \subseteq V$ such that for all $M, N \in V$*

$$\begin{array}{rcll} \Phi(M)(N) &=& \mathsf{U}^2_1, & \textit{if } M = N, \\ &=& \mathsf{U}^2_2, & \textit{else.} \end{array}$$

(v) *V is separable.*

[1] In Barendregt [1984] one uses this notation for a different system of numerals, denoted by $\ulcorner n \urcorner$, but that does not matter, as the $\mathbf{c}_n$ and the $\ulcorner n \urcorner$ are equivalent in the sense that for some combinators G, H one has $G\mathbf{c}_n = \ulcorner n \urcorner$ & $H\ulcorner n \urcorner = \mathbf{c}_n$.

One way to think of Böhm's theorem is that it says that separating morphisms can be realized by terms. Theorem 3.1 says that elements of a 'bounded' set of combinators can be uniformly mapped to their cut-off Böhm trees by a combinator. Theorem **??** shows that total morphisms are continuous with respect to the topology induced by the Böhm trees. Theorem 5.4 says how an extensional beta-eta morphism can be approximated by a term.

2. Preliminaries

2.1. NOTATION. (i) $\mathrm{BT}(M)$ is the *Böhm tree* of M (see Barendregt [1984] p. 212).

(ii) $\mathrm{BT}^q(M)$ is the *pruned Böhm tree of M at level q* (see Barendregt [1984] p. 219).

(iii) $M^{(q)}$ is the lambda term corresponding to $\mathrm{BT}^q(M)$.

(iv) $M[s]$ is the subterm of M rooted at the sequence number s (this is unlike the notation of Barendregt [1984] for typographical reasons).

(v) $\subseteq$ is the natural partial order of Böhm trees seen as partial functions.

(vi) We shall often identify finite Böhm trees with terms which they represent (replacing the symbol for bottom by the term Ω). As consequence of this we might say, for finite Böhm trees X, that $X(s)$ is undefined or equivalently $X(s)$ is unsolvable. We then also write $X \subseteq M$ for $X \sqsubseteq M$, which means $X \subseteq \mathrm{BT}(M)$.

(vii) Substitutions are treated using the substitution prefix of Curry and Feys [1968] p. 582, not the notation of Barendregt [1984] p. 27),

$$@ := [X_1/x_1, \ldots, X_n/x_n].$$

This denotes the result of simultaneously substituting X_i for x_i in M.

(viii) Sometimes a sequence of combinators can be specified by a list of numerical parameters and we do not want to make explict the enumeration. For example, a sequence $A_{i,j}$ which depends effectively on i and j has a representing combinator F such that $F\mathbf{c}_i\mathbf{c}_j = A_{i,j}$, but we may write simply $A[i,j]$ for $A_{i,j}$.

2.2. DEFINITION. A *V-set* is an RE set of combinators closed under beta (eta) conversion.

2.3. DEFINITION. Let V be a V-set.

(i) A V-morphism is a partial Ershov morphism whose domain includes V

(ii) A V-morphism f is V-representable if there exists an $F \in \Lambda^{\emptyset}$ such that

$$\forall M \in V. FM = f(M).$$

(iii) Δ is a *Church delta* for V if for all $M, N \in V$ one has

$$\begin{array}{lcll} \Delta MN & = & \mathsf{U}^2_1, & \text{if } M = N; \\ \Delta MN & = & \mathsf{U}^2_2, & \text{if } M \neq N. \end{array}$$

(iv) If M is in V then f is a V-test for M if whenever $N \in V$

$$\begin{aligned} f(N) &= \mathsf{U}^2_1, \quad \text{if } M = N, \\ f(N) &= \mathsf{U}^2_2, \quad \text{if } M \neq N. \end{aligned}$$

(v) The partial recursive function φ is said to *majorize* V if for each $M \in V$ whenever $BT(M)(s) = \langle w, n\rangle$ we have $\varphi(s)\downarrow$ and $\max\{\mathrm{lh}(w), n\} < \varphi(n)$.

(vi) V is said to be *majorizable* if there is a φ majorizing V. (Note; If φ majorizes V then we can w.l.o.g. assume that φ is monotone w.r.t. the extension ordering of finite sequences for we can always replace f by $\hat{\varphi}$ defined by

$$\hat{\varphi}(n_1, \dots, n_k) = \max\{\varphi(n_1, \dots, n_t) \mid 0 < t < k+1\}.$$

(vii) V is said to be *separable* if there is a combinator D such that

(1) $\forall M \in V.DM$ equals a Church numeral;

(2) $\forall M, N.[DM = DN \;\Leftrightarrow\; M = N]$.

(viii) If φ *majorizes* V we will define $BT^{\varphi}(M)$ for each $M \in V$. This $BT^{\varphi}(M)$ will be an infinite eta expansion of $BT(M)$, indeed it will satisfy the expansion condition: $BT^{\varphi}(M)(s)$ is an at most $\varphi(s) \cdot 4^{\mathrm{lh}(s)}$ eta expansion of $BT(M)(s)$. We fix in advance a list of free variables $v_1, \dots, v_p, \dots$ which will never occur in any terms below unless we put them there.

(ix) Write $B(p, q, r, s)$ for the set of possibly open terms X such that $BT^q(X)$ satisfies

(1) Free variables only from $v_1, \dots, v_p$.

(2) All lambda prefixes have length $< r$.

(3) Number of arguments (branching) $< s$.

We write $B(q, r, s)$ for $B(0, q, r, s)$. If $X \in B(p, q, r, s)$ then $X[p, q, r, s]$ is defined as follows

Case 1. $X = \lambda x_1 \dots x_n.v_i X_1 \dots X_m$. Then

$$X[p, q, r, s] := \lambda v_{p+1} \dots \lambda v_{p+n}.v_i @X_1[p+n, q-1, r, s] \dots @X_m[p+n, q-1, r, s],$$

where

$$@ := [v_{p+1}/x_1, \dots, v_{p+n}/x_n].$$

Case 2. $X = \lambda x_1 \dots x_n.x_i X_1 \dots X_m$. Then

$$X[p, q, r, s] := \lambda v_{p+1} \dots v_{p+n}.\lambda w_1 \dots w_{s+r-1-n}.v_i Y_1 \dots Y_m w_{s+r-1-n},$$

where

$$\begin{aligned} Y_k &:= @X_k[p+n, q-1, s+r, 2s+2r-2], \quad \text{for } 1 \le k \le m, \\ @ &:= [v_{p+1}/x_1, \dots, v_{p+n}/x_n]. \end{aligned}$$

Suppose that φ majorizes V. We define $BT^{\varphi}(M)$ for $M \in V$ as follows.

$$\mathrm{BT}^{\varphi}(M)(s) := M[0, \mathrm{lh}(s)+1, \varphi(s), \varphi(s)](s).$$

(x) V is said to be *extensional* if for all M, N one has

$$M \in V \ \&\ \mathrm{BT}(M) = \mathrm{BT}(N) \ \Rightarrow\ N \in V.$$

The V-morphism Φ is said to be *extensional* if for all $M, N \in V$

$$\mathrm{BT}(M) = \mathrm{BT}(N) \ \Rightarrow\ \mathrm{BT}(\Phi(M)) = \mathrm{BT}(\Phi(N)).$$

(xi) Φ is said to be *Scott continuous on* V if for each $M \in V$ and p there exists $X \subseteq \mathrm{BT}(M)$ such that for all $N \in V$

$$X \subseteq \mathrm{BT}(N) \ \Rightarrow\ \mathrm{BT}^p(\Phi(M)) \subseteq \mathrm{BT}(\Phi(N))$$

(xii) The V-morphism Φ which is Scott continuous on V is said to be *sequentially convergent on* V if for each p there exists a recursive enumeration $X_0, \ldots, X_m, \ldots$ of finite Böhm trees such that

(1) $\forall M \in V \exists X_i \subseteq \mathrm{BT}(M) \forall N \in V. X_i \subseteq \mathrm{BT}(N) \ \Rightarrow\ \mathrm{BT}^p(\varphi(M)) \subseteq \mathrm{BT}(\varphi(N))$.

(2) $X_i(s)\!\downarrow\ \Rightarrow\ X_{i+1}(s)\!\downarrow$.

Property (1) is called *convergence* and (2) is *sequentiality.* Such a sequence is called a *spread* for φ. A spread for φ may not be uniformly recursive in p.

(xiii) A term $M \in V$ is said to be *1-distinct* if there exists s such that for all $N \in V$ one has that

$$\mathrm{BT}(M)(s) \text{ and } \mathrm{BT}(N)(s)\!\downarrow \text{ and } M \neq N \ \Rightarrow$$
$$\Rightarrow\ \mathrm{BT}(M)(s) \text{ is distinct from } \mathrm{BT}(N)(s).$$

See Barendregt [1984] p. 253. A term $M \in V$ is $n+1$-*distinct* if there exists an s such that for all $N \in V$ the $\mathrm{BT}(N)(s)\!\downarrow$ and all the set

$$V - \{N \mid \mathrm{BT}(N)(s) \text{ is distinct from } \mathrm{BT}(M)(s)\} \text{ is n-distinct.}$$

We begin with some elementary facts about the definitions which answer some preliminary questions which might have occurred to the reader.

2.4. Facts. (i) The following statements (1) and (2) are equivalent.

(1) There is a V-morphism Φ such that
$\forall M, N \in V.[\Phi(M) = \Phi(N) \ \Rightarrow\ M = N]$ and
$\forall M \in V \exists n \in \mathbb{N}. \Phi(M) = \mathbf{c}_n$.

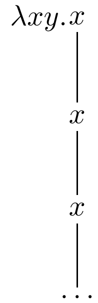
(2) $\{M \neq N \mid M, N \in V\}$ is RE.

(ii) If V is extensional then every extensional V-morphism extends to a total extensional V-morphism. This follows from Statman and Barendregt [1999] theorem 2.

(iii) There exists a V-set V with a V-test for each member of V but s.t. $\{M \neq N | M, N \in V\}$ is not RE. Define the partial recursive function φ such that

$$\begin{aligned} \varphi(e, x) &= x, && \text{if } \{e\}(e) \text{ converges in exactly x steps,} \\ &= 1 + \varphi(e, x+1), && \text{else.} \end{aligned}$$

By Kleene's theorem φ is represented by a lambda term F. Thus $F\mathbf{c}_n$ has finite Böhm tree $\mathrm{BT}(\mathbf{c}_n)$ if $\{e\}(e)$ converges in n steps and otherwise $F\mathbf{c}_e$ has the infinite Böhm tree



For each e we can construct e^* such that e^* exactly simulates e but $(e \neq e^*)$. Then $(Fe \neq Fe^*) \;\Leftrightarrow\; e(e)\!\uparrow$ (diverges).

(iv) If V is extensional with an extensional V-test for each member of V, or V is not extensional but there is a total extensional V-test for member of V then $\{M \neq N \mid M, N \in V\}$ is RE.

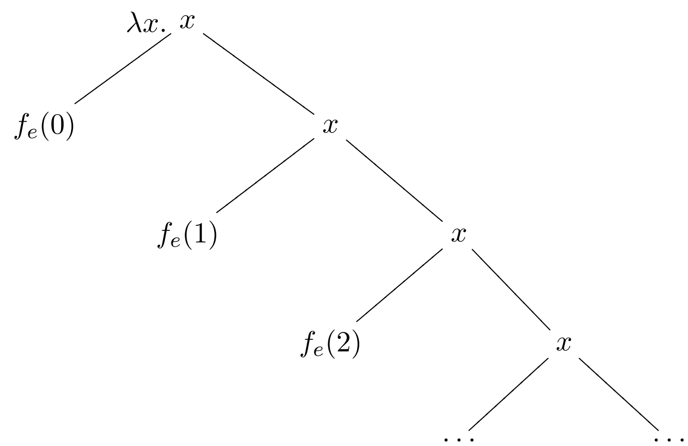
(v) There exists a V-set V such that each member of V has a V-representable V-test, $\{M \neq N | M, N \in V\}$ is RE, but there is no Church delta for V. First let T be Kleene's unique T predicate i.e. $T(e,x,y)$ & $T(e,x,z) \;\Rightarrow\; y = z$ and note that

$$\forall e \exists y \forall x [T(e,e,y) \vee \neg T(e,e,x)]$$

we define an enumeration of total recursive functions f_n by

$$\begin{array}{rcll} f_n(2^k(2e+1)) & = & e, & \text{if } n = e \text{ and } k = 0, \\ & = & e, & \text{if } n = e, k > 0, \text{ and } T(e,e,k), \\ & = & e, & \text{if } n < e, k = 0, \text{ and } T(e,e,n), \\ & = & e, & \text{if } e < n, k = 0, \text{ and } T(e,e,n-1), \\ & = & 0, & \text{else.} \end{array}$$

Now it is easy to define terms M_n such that the Böhm tree of M_e is



and let V be the beta closure of the set of all such M_n.

(vi) There exists a V-set V such that each member of V has a V-representable V-test but V is not majorizable.

3. A refinement of Böhm's theorem

3.1. THEOREM. *If V is majorized by f then there exists F such that*

$$\forall M \in V. FM = \mathrm{BT}^f(M).$$

PROOF. First we make the following definitions.

$$\begin{array}{rcl}
\$(q,r,s) & := & (\max\{r,s\} * 4^q) + 1. \\
\langle X_1,\ldots,X_n\rangle & := & \lambda a.aX_1\ldots X_n. \\
K[i,n] & := & \lambda x_1\ldots x_n.x_i. \\
K^*[i,n] & := & \lambda ax_1\ldots x_n b.ax_1\ldots x_{i-1}bx_{i+1}\ldots x_n. \\
A & := & \lambda xy.y(\lambda ab.\langle aK(b(x(aK^*+1)), aK^*+1\rangle)\langle \mathsf{I},0\rangle K. \\
Q & := & \lambda uvwxyz.\langle u,v,w,x,y,z\rangle. \\
R & := & \lambda uvwxyzabcdef.\langle u,b,c,d,e,f\rangle. \\
L & := & \lambda uvwxyzabcdef.\langle a,v,w,x,y,z\rangle. \\
J & := & \lambda u.u(\lambda abcdefg.f)\mathsf{IIIII}(\lambda abcdefg.g))\mathsf{IIIIII}. \\
\circ & := & \text{infix notation for } B := \lambda abc.a(bc)\text{, with association to the left.} \\
Y! & := & \text{Curry's paradoxical combinator.} \\
D & := & \lambda n.Y!(\lambda uvwxyz. \\
 & & \langle \lambda abcdef.Jau(\lambda pqrst.\langle R,p,q,r,s,t\rangle), v,w,x,y,z\rangle \circ AQn). \\
G & := & \lambda ij\lambda x_1\ldots x_i.\langle R,j,j,j,j,,j\rangle. \\
G & := & G00 = \langle R,0,0,0,0,0,\rangle. \\
H & := & \lambda ij.\langle L,0,0,0,0,0\rangle \circ A(KG)i \circ A(Gi)j. \\
V[i,j,k,m,n] & := & \lambda x_1\ldots x_k.\langle \lambda abcdef.fe, Dnijkm\rangle \circ (AQn) \circ \langle x_1,\ldots,x_k\rangle. \\
F & := & \langle V[p+1,r-1,s-1,\$(q,r,s),\$(q,r,s)],\ldots, \\
 & & V[p+r-1,r-1,s-1,\$(q,r,s),\$(q,r,s)]\rangle.
\end{array}$$

Next we construct an algorithm. The reader will easily be able to construct a term which executes the algorithm. It is convenient for purposes of exposition not to do this here but rather to argue about the algorithm directly.

Algorithm[2] $A(X \in \Lambda^{\emptyset}; p, q, r, s, j \in \mathbb{N})$.

$$
\begin{array}{llrcl}
(1) & \text{Set} & H & := & H(2s+2r-2)(s+r-1) \\
 & & h & := & (2s+2r-2)^{*}4^{q} \\
 & & F & := & Fpq(s+r)(2s+2r-1). \\
(2) & \multicolumn{4}{l}{\text{Reduce } HX \text{ to normal form.}} \\
(3) & \multicolumn{4}{l}{\text{If the 1st component of } HX \text{ is } R \text{ then go to (9) else continue.}} \\
(4) & \text{Set} & i & := & \text{2nd component of } HX \\
 & & H & := & H(h+2s+2r-1)(2s+2r-2). \\
(5) & \multicolumn{4}{l}{\text{Reduce } H(FX) \text{ to normal form.}} \\
(6) & \text{Set} & t & := & 2s+2r-2-(\text{2nd component of } H(FX)). \\
(7) & \text{Set} & \text{Output} & := & \#\langle \lambda v_{p+1} \dots v_{p+s+r-1}.v_i, t\rangle. \\
(8) & \text{Set} & X & := & AK*(2s+2r-2-t)(F(K^{*}[j, r-1]X)) \\
 & & & & K[j, h+2s+2r] \text{ and go to (12).} \\
(9) & \text{Set} & i & := & \text{2nd component of } HX \\
 & & j & := & \text{3rd component of } HX \\
 & & k & := & \text{4th component of } HX \\
 & & m & := & \text{5th component of } HX \\
 & & l & := & \text{6th component of } HX \\
 & & t & := & j+k+1-l.
\end{array}
$$

$$
\begin{array}{llrcl}
(10) & \text{Set} & \text{Output} & := & \#\langle \lambda v_{p+1} \dots v_{p+j} v_i, t\rangle. \\
(11) & \text{Set} & X & := & F(K^{*}[l, r-1]X)K[j, k+m+2]. \\
(12) & \text{Set} & p & := & p+s+r-1 \\
 & & q & := & q-1 \\
 & & r & := & s+r \\
 & & s & := & 2s+2r-1.
\end{array}
$$

$B^{*}(p, q, r, s)$ is the set of $X^{(}q)$ with $X \in B(p, q, r, s)$ after substituting $V(i, j, k, m, n)$ for all occurrences of $v_i \in X^{(}q)$ and the reducing the resulting redexes, where k, m, n can vary with i but must satisfy that

$$
\begin{array}{lrcl}
(a) & i & < & p+1, \\
(b) & k & < & s, \\
(c) & \$(q, r, s) & < & m+1, \\
(d) & \$(q, r, s) & < & n+1,
\end{array}
$$

(*e*) the number of arguments of any occurrence of $v_i < k+1$.
(*f*) The length of any lambda prefix followed by v_i as the head variable is j.

W.l.o.g. we may assume that X is $X^{(q)}$,and suppose that we have a corresponding member of $B^{*}(p, q, r, s)$. By eta expansions we can assume that every lambda prefix in this term has length $= s+r-1$ ($r-1$ from $B(p, q, r, s)$ and s from the $V(i, k, m, n)$).This should include those introduced at the immediate decendants of the root by these eta expansions but not those further below. Thus every variable originally in X has now at most $2s+2r-2$ arguments (with the head variable of $V(i, j, k, m, n)$ a notable exception in the general case) Let

[2] This means that A is the name of the algorithm and that X, p, q, r, s, j form its input.

X^* be the result of the substitutions, reductions and eta expansions.Now suppose that X^* is the input X in the algorithm. First suppose that X^* has the head normal form $\lambda y_1 \ldots y_{s+r-1}.y_i X_1^* \ldots X_t^*$, where $t < 2s+2r-1$. In this case the head variable of X is not one of the ones substituted for in the operation * and it is understood that some of the X_i^* may be the result of eta expansion at the head of X^*. Writing $n(s,r) = 2s+2r-2$ and just k for the Church's numeral $\mathbf{c}_k$ we compute as follows.

$$\begin{aligned} H(n(s,r))(s+r-1)X^* &= X^*(G(n(s,r))1)\ldots(G(n(s,r))(s+r-1))G\ldots \\ &\quad G\langle L,0,0,0,0,0\rangle \\ &= G(n(s,r))iZ_1\ldots Z_t G^{\sim n(s,r)}\langle L,0,0,0,0,0\rangle \\ &= \langle R,i,i,i,i,i\rangle G^{\sim t}\langle L,0,0,0,0,0\rangle \\ &= \langle R,i,i,i,i,i\rangle\langle L,0,0,0,0,0\rangle \\ &= \langle L,i,i,i,i,i\rangle \end{aligned}$$

and this is returned by line (2) of the algorithm. In addition, let

$$\begin{aligned} h &= \$(n(s,r),s+r,2s+2r-1)Fpn(s,r)(s+r)(2s+2r-1)X^* \\ &= V[p+i,s+r-1,n(s,r),h,h]U_1\ldots U_t \\ &= \lambda x_{t+1}\ldots\lambda x_{n(s,r)}.\langle U_1,\ldots,U_t,x_{t+1},\ldots,x_{n(s,r)},Q^{\sim h}(\lambda abcdef.fe), \\ &\quad Dh(p+i)(s+r-1)(n(s,r))h\rangle, \end{aligned}$$

where $U_i = X_i^{**}$ for the appropriate substitution **. So, writing $m(s,r,t) = n(s,r)-t$, we have

$$\begin{aligned} &H(h+n(s,r)+1)(n(s,r))(Fpq(s+r)(n(s,r)+1)X^*) = \\ &= G(h+n(s,r)+1)(m(s,r,t))W_1\ldots W_t(G(h+n(s,r)+1)1)\ldots \\ &\quad (G(h+n(s,r)+1)(s+2r-2-t-1))Q^{\sim h}(\lambda abcdef.fe) \\ &\quad (Dh(p+i)(s+r-1)(n(s,r))h)(G(h+n(s,r)+1)(n(s,r)+1-t))\ldots \\ &\quad (G(h+2s+2r)(n(s,r)+1)))G^{\sim n(s,r)+1}\langle L,0,0,0,0,0\rangle \\ &= \langle R,m(s,r,t),m(s,r,t),m(s,r,t),m(s,r,t),m(s,r,t)\rangle \\ &\quad G^{\sim n(s,r)+1}\langle L,0,0,0,0,0\rangle \\ &= \langle L,m(s,r,t),m(s,r,t),m(s,r,t),m(s,r,t),m(s,r,t)\rangle \end{aligned}$$

and this is returned by line 5 of the algorithm. Also

$$\begin{aligned} &AK^*(m(s,r,t))(Fpq(s+r)(n(s,r)+1)(K^*[j,r-1]X))K[j,h+2s+2r] = \\ &= (Fpq(s+r)(n(s,r)+1)(K^*[j,r-1]X)\mathsf{I}^{\sim m(s,r,t)}K[j,h+2s+2r] \\ &= X_j^{**} \end{aligned}$$

and $X_j^{**} \in B^*(p+s+r-1,q-1,s+r,n(s,r)+1)$, for the appropriate substitution **, and this is set equal to X by line 8 of the algorithm. Next suppose X has as a head variable one of the variables, say the i-th, substituted for.Let X^* have the head normal form

$$\begin{aligned} \lambda y_1\ldots y_j\lambda x_{t+1}\ldots x_k\lambda a\lambda z_1\ldots z_l.\quad & aX_1^*\ldots X_t^* x_{t+1}\ldots x_k Q^{\sim m}(\lambda abcdef.fe) \\ & (Dnijkm)z_1\ldots z_l, \end{aligned}$$

where $l+k-t+j+1=s+r-1$, for $t<s$. Then, writing
$u=j+k-t+1, v=j+t+k$

$$\begin{aligned}
& H(n(s,r))(s+r-1)X^* = \\
= \;& G(n(s,r))(u)Z_1 \ldots Z_t \\
& (G(n(s,r))(j+t+1)) \ldots (G(n(s,r))(v))Q^{\sim m} \\
& (\lambda abcdef.fe)(Dnijkm) \\
& (G(n(s,r))(v+1)) \ldots (G(n(s,r))(s+r-1))) \\
& G^{\sim n(s,r)}\langle L,0,0,0,0,0\rangle \\
= \;& \langle R,u,u,u,u,u\rangle Q^{\sim m-(n(s,r)-k-t)}(\lambda abcdef.fe)(Dnijkm) \\
& (G(s+2r-1)(v+1)) \ldots (G(s+2r-1)(s+2r-1))) \\
& G^{\sim n(s,r)}\langle L,0,0,0,0,0\rangle \\
= \;& \langle R,u,u,u,u,u\rangle(\lambda abcdef.fe)(Dnijkm) \\
& (G(n(s,r))(v+1)) \ldots (G(n(s,r))(s+r-1)) \\
& G^{\sim n(s,r)}\langle L,0,0,0\rangle \\
= \;& (D(ijkm(u))(G(n(s,r))(v+1)) \ldots (G(s+2r-1)(s+r-1)) \\
& G^{\sim n(s,r)}\langle L,0,0,0,0,0\rangle \\
= \;& (Dnijkm(u))G^{\sim n(s,r)}\langle L,0,0,0,0,0\rangle \\
= \;& (Dnijkm(u)\langle L,0,0,0,0,0\rangle \\
= \;& \langle R,i,j,k,m,u\rangle
\end{aligned}$$

and this is returned by line 2 of the algorithm. Also

$$Fpq(s+r)(n(s,r)+1)(K^*[l,r-1]X)K[j,k+m+2] = X_j^{**}$$

and $X_j^{**} \in B^*(p+s+r-1, q-1, s+r, n(s,r)+1)$ for the appropriate substitution ** , and this is set equal to X by line 11 of the algorithm. Finally suppose that X is unsolvable. Then the algorithm returns unsolvable. We now iterate the algorithm as follows.

Iterated Algorithm $A^*(f, M, n_1, \ldots, n_k)$.

(1) Set
$$\begin{aligned}
p &:= 0 \\
q &:= l \\
r &:= f(n_1, \ldots, n_k) \\
s &:= f(n_1, \ldots, n_k) \\
X &:= M
\end{aligned}$$

(2) For $i=1,\ldots,k$ do Set $j=n_i$; A(X).

The following should be clear from the previous discussion.
Claim. If BT$(M)(n_1,\ldots,n_k)\downarrow$, then A^* yields BT$^f(M)(n_1,\ldots,n_k)$. ∎

4. Böhm's theorem for V-sets

4.1. Theorem. *For V an infinite V-set the following are equivalent.*

(i) *There are combinators $S, P, 0, \text{ZERO}_?$, such that $(V, S, P, O, \text{ZERO}_?)$ is an numeral system with predecessor P, see Barendregt [1984] Proposition 6.4.3 and the remark following.*

(ii) *Every V-morphism is V-representable.*

(iii) *There is a Church's delta for V.*

(iv) *There is a* V*-morphism* Φ *such that*

$$\forall M \in V. \Phi(M) \text{ is a } V\text{-test for } M.$$

(v) V *is separable.*

PROOF. We shall prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Write $\mathbf{v}_n := S^n O$. By Barendregt [1984] Lemma 6.4.5 there exists an H such that

$$H(\mathbf{v}_n) = \mathbf{c}_n. \tag{1}$$

The function $g(n) = \#\mathbf{v}_n$ is total recursive, so by Barendregt [1984] Theorem 6.4.3, it is lambda definable w.r.t. $(V, S, P, O, \text{ZERO}_?)$ by, say, G, i.e.

$$G\mathbf{v}_n = \mathbf{v}_{g(n)} = \mathbf{v}_{\#\mathbf{v}_n}. \tag{2}$$

Now suppose Φ is a partial morphism whose domain contains the set V. Then Φ extends to a total morphism $\overline{\Phi}$, by Statman and Barendregt [1999] Theorem 2. By definition there is a total recursive function f, such that

$$\overline{\Phi}(M) = \mathsf{E}\mathbf{c}_{f(\#M)}. \tag{3}$$

This f is lambda definable on $(V, S, P, O, \text{ZERO}_?)$ by, say, F. This means that

$$F\mathbf{v}_n = \mathbf{v}_{f(n)}. \tag{4}$$

Let E be Kleene's enumerator and set $J := \lambda x.\mathsf{E}(H(F(Gx)))$. Then

$$\begin{aligned} J(\mathbf{v}_n) &= E(H(F(G(\mathbf{v}_n)))) \\ &= E(H(F(\mathbf{v}_{\#\mathbf{v}_n}))), && \text{by (2)}, \\ &= E(H(\mathbf{v}_{f(\#\mathbf{v}_n)}))), && \text{by (4)}, \\ &= E\mathbf{c}_{f(\#\mathbf{v}_n)}, && \text{by (1)}, \\ &= \overline{\Phi}(\mathbf{v}_n), && \text{by (3)}, \\ &= \Phi(\mathbf{v}_n). \end{aligned}$$

Thus Φ is V-represented by J.

(2) $\Rightarrow$ (3). Let $M, N \in V$ with $M \neq N$. Define a partial morphism Φ by

$$\begin{aligned} L = M \Rightarrow \Phi(L) &= \mathsf{U}^2_1 \\ L = N \Rightarrow \Phi(L) &= \mathsf{U}^2_2. \end{aligned}$$

Then this partial morphism extends to a total morphism by Statman and Barendregt [1999] Theorem 2, which is *a fortiori* a V-morphism. By hypothesis this morphism is V-representable and thus for some F one has

$$FM = \mathsf{U}^2_1 \;\&\; FN = \mathsf{U}^2_2.$$

Hence, in particular, the set $\{(M, N) \in V^2 \mid M \neq N\}$ is RE. Thus the partial function Φ on V^2 such that

$$\begin{aligned} \Phi(M, N) &= \mathsf{U}^2_1, && \text{if } M = N, \\ \Phi(M, N) &= \mathsf{U}^2_2, && \text{if } M \neq N, \end{aligned}$$

is a V-morphism, which by hypothesis is representable. In conclusion, there is a Church delta for V.

(3) $\Rightarrow$ (4). Immediate.

(4) $\Rightarrow$ (5). Let Φ be as in (4) and let F λ-define its representing function φ. Let G be an enumeration of the Gödel numbers of members of V. By the Fixedpoint Theorem let

$$Dxy = (\mathsf{E}(F(Gy))x(Gy)(Dx(x+y)))\mathbf{c}_0.$$

This gives (5).

(5) $\Rightarrow$ (1). The set $\{(M,N)\in V^2 \mid M \neq N\}$ is (after coding) RE by separability. Consider the following procedure. Enumerate V, distributing the elements of V into beta equivalence classes as they are enumerated. Send all the elements in the nth class to the first element in the $n+1$-st class. This procedure defines a partial recursive function f whose domain includes the set of Gödel numbers of members of V. The function f is lambda defined by a term F. If in this procedure we replace $n+1$ by $\mathrm{pred}(n)$ we get another similar partial recursive g lambda defined by G. Let O be any member of the first equivalence class and put $S = \lambda x.\mathsf{E}(F(Dx)), P = \lambda x.\mathsf{E}(G(Dx)), \mathrm{Zero}_? = \lambda x.Z_?(DO)(Dx)$, where E is Kleene's enumerator and $Z_?$ is the test for zero for Church numerals. This gives us a numeral system with predecessor. ∎

It immediately follows that not every total Ershov morphism on $\Lambda^{\emptyset}$ is representable. Indeed, by Theorem 3.1 it would follow that there is a Church's delta Δ for $\Lambda^{\emptyset}$, but then $\lambda x.\Delta x \mathsf{U}^2_2$ has no fixed-point.

5. Approximation on majorizable V sets

5.1. Lemma. *Suppose that Φ is an extensional total beta(eta) morphism. Then Φ is monotone.*

Proof. Suppose not. Then there exists M, N such that $\mathrm{BT}(M) \subseteq \mathrm{BT}(N)$ but there exists an s such that one of the two following statements hold.

(a) $\mathrm{BT}(\Phi(M))(s)\downarrow$ and $\mathrm{BT}(\Phi(N))(s)\downarrow$ are distinct;

(b) $\mathrm{BT}(\Phi(M))(s)\downarrow$ but $\mathrm{BT}(\Phi(N))(s)\uparrow$.

For each e we define an RE Böhm tree $T(e)$ by enumeration as follows. At stage k

(i) If $\{e\}(e)$ has not converged in $< k$ steps then do k steps in the enumeration of $\mathrm{BT}(M)$.

(ii) If $\{e\}(e)$ has converged in $l < k$ steps then do k steps in the enumeration of $\mathrm{BT}(N)$. By Barendregt [1984] Theorem 10.1.23 we can effectively calculate a combinator $N(e)$ such that $\mathrm{BT}(N(e)) = T(e)$. Now we have either case (a) or (b). In the first case

$\{e\}(e)\downarrow \Leftrightarrow \mathrm{BT}(\Phi(M))(s), \mathrm{BT}(\Phi(N(e)))(s)$ are distinct.

Since s does not depend on e this solves the halting problem, which is impossible. Therefore we have case (b) and hence

$\{e\}(e)\uparrow \Leftrightarrow \mathrm{BT}(\Phi(N(e)))(s)$ is defined

$\{e\}(e)\downarrow \Leftrightarrow \mathrm{BT}(\Phi(N(e)))(s)$ is undefined

which also solves the halting problem. ∎

5.2. THEOREM. *If Φ is an extensional total beta(eta) morphism, then Φ is continuous w.r.t the Scott topology*

PROOF. Let $X \subseteq \mathrm{BT}(\Phi(M))$. Suppose towards a contradiction that for each $Y \subseteq \mathrm{BT}(M)$ there exists N such that $Y \subseteq N$ but not $X \subseteq \mathrm{BT}(\Phi(N))$. Now by monotonicity there cannot be a finite Böhm tree Y such that $Y \subseteq \mathrm{BT}(M)$ and $X \subseteq \mathrm{BT}(\Phi(Y))$. For each e we define an RE Böhm tree $T(e)$ by enumeration as follows. At stage k do the following.

1. If $\{e\}(e)$ has not converged in $< k$ steps then do k steps in the enumeration of $\mathrm{BT}(M)$.
2. If $\{e\}(e)$ has converged in $l < k$ steps then go to step $k+1$.

By Barendregt [1984] Theorem 10.1.23 we can effectively calculate a combinator $N(e)$ such that $\mathrm{BT}(N(e)) = T(e)$. Now we have either
$\{e\}(e)\uparrow \;\Leftrightarrow\; X \subseteq \mathrm{BT}(\Phi(N(e)))$
$\{e\}(e)\downarrow \;\Leftrightarrow\;$ every finite approximation Z of $\mathrm{BT}(\Phi(N(e)))$ has $X \not\subseteq Z$.
Hence we can recursively decide the halting problem. ■

5.3. NOTATION. If T, S are Böhm-trees, write $T \subseteq S(\mathrm{mod}\ \eta)$ iff for some finite η-expansions T', S' of T, S respectively, one has $T' \subseteq S'$.

5.4. THEOREM. *Suppose V is majorizible and Φ is a total extensional morphism sequentially convergent on V. Then*

$$\forall p \in \mathbb{N} \exists F \forall M. \mathrm{BT}^p(\Phi(M)) \subseteq \mathrm{BT}(FM)(\mathrm{mod}\ \eta).$$

PROOF. Suppose that V is majorized by φ. By Theorem 4.1 there exists G such that $GM = \mathrm{BT}^{\varphi}(M)$ for all $M \in V$. Since Φ is sequentially convergent there exists a recursive enumeration of finite Böhm trees $X_0, \ldots, X_m, \ldots$ as described in Definition 2.3(xii). We define a new algorithm approximating Φ.

Approximation Algorithm $A(M \in \Lambda^{\emptyset})$.

(1) Set $m := 0$.
(2) If $X_m \subseteq GM$ then set Output $:= E(\Phi(X_m))$ else
set $m := m+1$ and repeat. ■

5.5. COROLLARY. *Suppose V is majorizable. Then for all $M \in V$*

there is a V-representable V-test for M $\Leftrightarrow$ $\exists m. M$ is m-distinct.

PROOF. ($\Rightarrow$) Suppose that V is majorizable and F represents a V-test for M. Since F is Scott continuous there exists X such that $X \subseteq \mathrm{BT}(M)$ and for all N, whether in V or not, $X \subseteq \mathrm{BT}(N) \;\Rightarrow\; FN = \mathsf{U}^2_1$. Let X be minimal with this property. In particular, if $N \in V$ then there exists a finite Böhm tree $Y := Y_N$ such that $Y \subseteq \mathrm{BT}(N)$ and $FY = K^*$ then $F(X/\!\!\backslash Y) \subseteq \mathsf{U}^2_1$ and $F(X/\!\!\backslash Y) \subseteq \mathsf{U}^2_2$. Thus $F(X/\!\!\backslash Y)$ is unsolvable. Now by the sequentiality theorem Barendregt [1984] 14.4.8 we must have X and Y distinct. Recall that if s is a sequence then $X[s]$ is the subtree of X rooted at s. We now show that the sequences s such that $X[s]$ is solvable can be totally ordered by a relation R such that (1)if s extends t then tRs (2) for each $Y := Y[N]$ there exists s such that $X(s)$ and $Y(s)$ are distinct and $Y[t]$ is solvable for all tRs. Toward this end let $U = \lambda a.Faxy$. By the standardization theorem there exists a head reduction $UX \twoheadrightarrow_h x$. We trace the occurrences of substitution instances of subterms $@X[s]$ of X which come to the head of this reduction (consolidating reductions of the same lambda prefix into a single "multi" step). We copy this reduction to a reduction of UY where $Y := Y[N]$ by the replacement proceedure * defined as follows

$$\begin{aligned} X[s]^* &= Y[s][U_1/x_1, \ldots, U_n/x_n]^* \\ &= [U_1^*/x_1, \ldots, U_n^*/x_n](@X[s])^* \\ &= @^*Y[s]^*. \end{aligned}$$

Now the ordering R is defined by $tRs \;\Leftrightarrow\; @_1X[t]$ comes to the head before $@_2X[s]$ for some $@_1$ and all $@_2$. Clearly R has the property (1). Now in the copied reduction there must come some step where copying fails to produce a redex reductum pair for otherwise we would have the head reduction

$$UY \twoheadrightarrow_h x,$$

but we know that $UY \twoheadrightarrow_h y$. At this step we must have on the X side $(@X[s])U_1 \ldots U_n$ and on the Y side $(@^*Y[s])U_1^* \ldots U_n^*$. Thus $X[s]$ and $Y[s]$ are distinct at the root. Thus R has the property (2). It follows that M is $|X|$-distinct. ($\Leftarrow$) By Theorem 5.4. ■

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