THEORETICAL PEARLS

Applications of Plotkin-terms: partitions and morphisms for closed terms

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Abstract

This theoretical pearl is about the closed term model of pure untyped lambda-terms modulo β-convertibility. A consequence of one of the results is that for arbitrary distinct combinators (closed lambda terms) M, M', N, N' there is a combinator H such that

$$HM = HM' \neq HN = HN'.$$

The general result, which comes from Statman [1998], is that uniformly r.e. partitions of the combinators, such that each "block" is closed under β-conversion, are of the form $\{H^{-1}\{M\}\}_{M\in\Lambda^\emptyset}$. This is proved by making use of the idea behind the so-called Plotkin-terms, originally devised to exhibit some global but non-uniform applicative behavior. For expository reasons we present the proof below. The following consequences are derived: a characterization of morphisms and a counter-example to the perpendicular lines lemma for β-conversion.

1. Introduction

We use notations from recursion theory and lambda calculus, see Rogers [1987] and Barendregt [1984].

NOTATION. (i) φ_e is the e-th partial recursive function of one argument.

(ii) $W_e = \mathrm{dom}(\varphi_e) \subseteq \mathbb{N}$ is the r.e. set with index e.

(iii) Λ is the set of lambda-terms and $\Lambda^\emptyset$ is the set of closed-lambda terms (combinators).

(iv) $\mathcal{W}_e = \{M \in \Lambda^\emptyset \mid \#M \in W_e\} \subseteq \Lambda^\emptyset$; here $\#M$ is the code of the term M.

1.1. DEFINITION. (i) Inspired by Visser [1980] we define a *Visser-partition* (V-partition) of $\Lambda^\emptyset$ to be a family $\{\mathcal{W}_e\}_{e\in S}$ such that

(1) $S \subseteq \mathbb{N}$ is an r.e. set
(2) $\forall e{\in}S\, \forall M,N\ (M{\in}\mathcal{W}_e\ \&\ N = M) \Rightarrow N{\in}\mathcal{W}_e$.
(3) $\mathcal{W}_e \cap \mathcal{W}_{e'} \neq \emptyset \Rightarrow \mathcal{W}_e = \mathcal{W}_{e'}$.

(ii) A family $\{\mathcal{W}_e\}_{e\in S}$ is a *pseudo*-V-partition if it satisfies just 1 and 2.

1.2. DEFINITION. Let $\{\mathcal{W}_e\}_{e\in S}$ be a V-partition.

1. The partition is said to be *covering* if $\bigcup_{e\in S}\mathcal{W}_e = \Lambda^{\emptyset}$.
2. The partition is said to be *inhabited* if $\forall e{\in}S\ \mathcal{W}_e \neq \emptyset$.
3. A V-partition $\{\mathcal{W}_e\}_{e\in S'}$ is said to be *(extensionally) equivalent* with $\{\mathcal{W}_e\}$ if these families define the same collection of non-empty sets, i.e. if

$$\{\mathcal{W}_e \mid e{\in}S\ \&\ \mathcal{W}_e \neq \emptyset\} = \{\mathcal{W}_e \mid e{\in}S'\ \&\ \mathcal{W}_e \neq \emptyset\}.$$

1.3. EXAMPLE. Let H be some given combinator. Define

$$\mathcal{W}_{e(M,H)} = \{N{\in}\Lambda^{\emptyset} \mid HN = HM\},$$

Then $\{\mathcal{W}_e\}_{e\in S_H}$, with $S_H = \{e(M,H) \mid M{\in}\Lambda^{\emptyset}\}$, is an example of a covering and inhabited V-partition. We denote this V-partition by $\{\mathcal{W}_{e(M,H)}\}_{M\in\Lambda^{\emptyset}}$.

1.4. PROPOSITION. (i) *Every V-partition is effectively equivalent to an inhabited one.*

(ii) *Every V-partition can effectively be extended to a covering one.*

PROOF. (i) Given $\{\mathcal{W}_e\}_{e\in S}$ define $S' = \{e{\in}S \mid \mathcal{W}_e \neq \emptyset\}$. Then $\{\mathcal{W}_e\}_{e\in S'}$ is the required modified partition.

(ii) Given $\{\mathcal{W}_e\}_{e\in S}$ define

$$\mathcal{W}_{e(M)} = \{N \mid N = M \vee \exists e{\in}S\ M,N{\in}\mathcal{W}_e\}.$$

Then $\{\mathcal{W}_{e(M)}\}_{M\in\Lambda^{\emptyset}}$ is the required V-partition. ∎

The main theorem comes in two version. The second more sharp version is needed for the construction of so called inevitably consistent equations, see Statman [1999].

1.5. THEOREM (Main theorem). (i) *Let $\{\mathcal{W}_e\}_{e\in S}$ be a V-partition. Then one can construct effectively a combinator H such that for all $M,N{\in}\Lambda^{\emptyset}$*

$$HM = HN \Leftrightarrow M = N \vee \exists e{\in}S\ M,N{\in}\mathcal{W}_e. \qquad (*)$$

The construction of H is effective in the code of the underlying r.e. set S.

(ii) *Let $\{\mathcal{W}_e\}_{e\in S}$ be a pseudo-V-partition. Then one can construct effectively a combinator H such that if $\{\mathcal{W}_e\}_{e\in S}$ is an actual V-partition, then $(*)$ holds.*

The theorem will be proved in §2. It has several consequences. In order to state these we have to formulate the notion of morphism on $\Lambda^{\emptyset}$ and the so-called perpendicular lines lemma.

1.6. DEFINITION. Let $\varphi : \Lambda^{\emptyset} \to \Lambda^{\emptyset}$ be a map. Then φ is a *morphism* if

1. $\varphi(M) = \mathsf{Ec}_{f(\#M)}$, for some recursive function f.
2. $M = N \Rightarrow \varphi(M) = \varphi(N)$.

1.7. Lemma. (i) *Let F be a combinator and define $\varphi_H(M) \equiv HM$. Then φ_H is a morphism.*

(ii) *Let F, G be combinators such that for all $M \in \Lambda^\emptyset$ there exists a unique $N \in \Lambda^\emptyset$ with $FM = GN$. Then there is a map $\varphi_{F,G}$ such that $FM = G\varphi_{F,G}(M)$, for all M, which is a morphism.*

Proof. (i) For the coding $\#$ let app be the recursive function such that $\#(PQ) = \text{app}(\#P, \#Q)$. Define $f(m) = \text{app}(\#H, m)$. Then $\varphi_H(M) = \mathsf{E}\mathbf{c}_{f(\#M)}$. It is obvious that φ_H preserves β-equality.

(ii) Let $R(m,n)$ be an r.e. relation. The we have $R(m,n) \Leftrightarrow \exists z\ T(m,n,z)$, for some recursive T. Let $< n, z >$ be a recursive pairing with recursive inverses $< n, z > .0 = n, < n, z > .1 = z$. Define ($\mu$ is the least number operator)

$$\iota_n.R(m,n) = (\mu p.T(m, p.0, p.1)).0.$$

Then $\exists n \in \mathbb{N} R(m,n) \Rightarrow R(m, \iota_n.R(m,n))$. In order to construct the morphism $\varphi_{F,G}$, define

$$f(m) = \iota_n.F(\mathsf{E}\mathbf{c}_m) = G(\mathsf{E}\mathbf{c}_n).$$

By the assumption (existence) f is total. Define $\varphi_{F,G}(M) = \mathsf{E}\mathbf{c}_{f(\#M)}$. Now $f(\#M) = n \Rightarrow F(\mathsf{E}\mathbf{c}_n) = G(\mathsf{E}\mathbf{c}_n)$. Therefore $FM = G\varphi_{F,G}(M)$, for all M. The condition

$$M = M' \Rightarrow \varphi_{F,G}(M) = \varphi_{F,G}(M')$$

holds by the assumption (unicity). ■

One may wonder whether dropping the unicity condition in lemma 1.7 (ii) one may obtain a morphism by making a right uniformization. This is not the case.

1.8. Proposition. *There exists combinators F, G such that $\forall M \exists N\ FM = GN$ but without any morphism satisfying $\forall M\ FM = G\varphi(N)$.*

Proof. Let $\Delta = \mathsf{Y}\Omega$ and define $F = \lambda x.\langle x, \Delta, \mathsf{I}\rangle$ and $G = \lambda y.\langle \mathsf{E}y, y\Omega\Delta, y\mathsf{I}\rangle$. Then, see Statman [1986],

$$FM =_\beta GN \Leftrightarrow (N =_\beta c_n \vee N =_\beta \mathsf{I})\ \&\ \mathsf{E}N =_\beta M. \tag{1}$$

Any morphism φ such that $FM = G\varphi(M)$ would solve the convertibility problem recursively: one has by (1)

$$M = M' \Leftrightarrow \varphi(M) = \varphi(M'), \tag{2}$$

and since $\varphi(M), \varphi(M')$ have nf's by (1), the RHS of (2) is decidable. ■

1.9. Proposition. *Not every morphism is of the form φ_H.*

Proof. Let $F, G \in \Lambda^\emptyset$ be such that $F \circ G = \mathsf{I}$. Then F, G determine a so-called *inner model* $[\ ] = [\ ]^{F,G}$ as follows.

$$\begin{aligned} [x] &= x; \\ [PQ] &= F[P][Q]; \\ [\lambda x.P] &= G(\lambda x.[P]). \end{aligned}$$

Using the condition on F, G it can be proved that

$$M =_\beta N \;\Rightarrow\; [M] = [N].$$

Therefore defining $\varphi(M) = [M]$ we obtain a morphism.

Now take $F \equiv \lambda y.u\mathsf{I}$, $\Gamma \equiv \lambda xy.yx$. Then indeed $F \circ G = \mathsf{I}$ and for the resulting inner model one has $[I] = \lambda y.y\mathsf{I}$ and $[\Omega] = (\lambda y.y(\lambda z.z\mathsf{I}z))\mathsf{I}(\lambda y.y(\lambda z.z\mathsf{I}z))$.

Suppose towards a contradiction that the resulting φ is of the form φ_H. Then $H\mathsf{I} = \lambda y.y\mathsf{I}$, so H is solvable and hence has a hnf $\lambda x_1 \ldots x_n.x_i M_1 \ldots M_m$. But $H\Omega = (\lambda y.y(\lambda z.z\mathsf{I}z))\mathsf{I}(\lambda y.y(\lambda z.z\mathsf{I}z))$, which is unsolvable. Therefore the head-variable x_i is x_1. But then $H\Omega = \lambda x_2 \ldots x_n.\Omega M_1^* \ldots M_m^*$ which is not of the correct form. ∎

The following is a corollary to the main theorem.

1.10. Corollary. *Every morphism φ is of the form $\varphi_{F,G}$.*

Proof. Let φ be a given morphism. Define

$$\mathcal{W}_{e(N)} = \{Z \mid \exists M{\in}\Lambda^{\emptyset}\, [\varphi(M) = N \;\&\; [Z = \langle \mathbf{c}_0, M\rangle \;\vee\; Z = \langle \mathbf{c}_1, N\rangle]]\}.$$

Then $\{\mathcal{W}_{e(N)}\}$ is a V-partition. By the main theorem there exists an H such that

$$\begin{aligned} H\langle \mathbf{c}_0, M\rangle = H\langle \mathbf{c}_1, N\rangle &\Leftrightarrow \langle \mathbf{c}_0, M\rangle = \langle \mathbf{c}_1, N\rangle \vee N = \varphi(M) \\ &\Leftrightarrow N = \varphi(M). \end{aligned}$$

Define

$$\begin{aligned} F &= \lambda m.H\langle \mathbf{c}_0, m\rangle; \\ G &= \lambda n.H\langle \mathbf{c}_1, n\rangle. \end{aligned}$$

Then $FM = GN \;\Leftrightarrow\; N = \varphi(M)$. Therefore $\varphi = \varphi_{F,G}$. ∎

Note that for a given morphism φ one can define by

$$\mathcal{W}_{e(M,\varphi)} = \{N{\in}\Lambda^{\emptyset} \mid \varphi(M) = \varphi(N)\}.$$

This is an inhabited V-partition. It is not difficult to show that that each V-partition is equivalent to one of the form $\{\mathcal{W}_{e(M,\varphi)}\}$. Note that $\{\mathcal{W}_{e(M,H)}\} = \{\mathcal{W}_{e(M,\varphi_H)}\}$, see lemma 1.7. The following result shows that covering V-partitions are always of this more restricted form.

1.11. Corollary. *If $\{\mathcal{W}_e\}$ is a covering V-partition, then $\{\mathcal{W}_e\}$ is equivalent to $\{\mathcal{W}_{e(M,H)}\}_{M\in\Lambda^{\emptyset}}$ for some H, effectively found from $\{\mathcal{W}_e\}$.*

Proof. Let H be the combinator constructed effectively from $\{\mathcal{W}_e\}$. We will show that $\mathcal{W}_{e(M,H)} = \{N \mid HN = HM\}$ is equivalent to $\{\mathcal{W}_e\}$. Claim. For $N{\in}\mathcal{W}_e$ one has $\mathcal{W}_e = \mathcal{W}_{e(M,H)}$. Indeed,

$$\begin{aligned} N{\in}\mathcal{W}_e &\Leftrightarrow M = N \;\vee\; M, N{\in}\mathcal{W}_e \\ &\Leftrightarrow HN = HM \\ &\Leftrightarrow N{\in}\mathcal{W}_{e(M,H)}. \end{aligned}$$

Therefore, noting that $M\in\mathcal{W}_{e(M,H)}$,

$$\{\mathcal{W}_e \mid M\in\Lambda^\emptyset, \mathcal{W}_e \neq \emptyset\} \subseteq \{\mathcal{W}_{e(M,H)} \mid \mathcal{W}_{e(M,H)} \neq \emptyset, M\in\Lambda^\emptyset\}.$$

The converse inclusion holds also, since every M belongs to some $\mathcal{W}_e$ and hence $\mathcal{W}_{e(M,H)} = \mathcal{W}_e$ for this e. ■

The following theorem states that if a combinator, seen as function of n arguments, is constant—modulo Böhm-tree equality—on n perpendicular lines, then it is constant everywhere.

1.12. THEOREM (Perpendicular lines lemma). *Let F be a combinator. Suppose that for $n\in\mathbb{N}$ there are combinators M_{ij}, $1 \leq i \neq j \leq n$, and $N_1, \dots, N_n$ such that for all terms $Z\in\Lambda$ one has ($\cong$ denotes Böhm-tree equality, i.e. $M \cong N \Leftrightarrow BT(M) = BT(N)$)*

$$\begin{array}{lllllllll}
F & Z & M_{12} & \dots & M_{1n-1} & M_{1n} & \cong & N_1; \\
F & M_{21} & Z & \dots & M_{2n-1} & M_{2n} & \cong & N_2; \\
 & & & \dots & & & & \\
 & & & \dots & & & & \\
F & M_{n1} & M_{n2} & \dots & M_{nn-1} & Z & \cong & N_n.
\end{array}$$

Then for all $P_1, \dots, P_n\in\Lambda^\emptyset$ one has

$$FP_1 \dots P_n \cong N_1 (\cong N_2 \cong \dots \cong N_n).$$

PROOF. This is proved in Barendregt [1984], theorem 14.4.12. ■

The perpendicular lines lemma also holds for closed terms (i.e. the Z range over $\Lambda^\emptyset$). This is proved by Bethke [1999], who obsreved that Berry's sequentiality result, see Barendregt [1984] theorem 14.4.8, remains valid if in definition 14.4.2 of the notion "*is caused by*" the implication

$$M_i{}'|\beta = z \;\Rightarrow\; C[\vec{M}']|\alpha \neq \bot]$$

is replaced by

$$M_i{}'|\beta \neq \bot \;\Rightarrow\; C[\vec{M}']|\alpha \neq \bot].$$

A conjecture in Barendregt [1984] states that the perpendicular lines lemma with $\cong$ replaced by $=_\beta$ is correct for open terms. We do believe that this can be proved using using a result of Diderik van Daalen in *loc. cit.* exercise 15.4.8.

The following result shows that both changes (that is, for closed terms modulo β-conversion) make the perpendicular lines lemma invalid.

1.13. PROPOSITION. *If the perpendicular lines lemma is restricted to closed terms and if $\cong$ is replaced by $=_\beta$, then the perpendicular lines lemma is false for any $n > 1$.*

PROOF (For $n = 1$ the lemma is trivially true for $=_\beta$.). Let $n > 1$. For notational simplicity we assume $n = 2$ and give a counter example. Define

$$\begin{array}{lll}
\mathcal{W}_{e_1} & = & \{N\in\Lambda^\emptyset \mid N = \langle \mathsf{S}, \mathsf{S}\rangle\} \\
\mathcal{W}_{e_2} & = & \{N\in\Lambda^\emptyset \mid \exists Z\in\Lambda^\emptyset [N = \langle \mathsf{I}, Z\rangle \;\vee\; N = \langle Z, \mathsf{I}\rangle]\}
\end{array}$$

Then $\{\mathcal{W}_e\}_{e\in\{e1,e2\}}$ is a V-partition. Let H be the combinator obtained from this partition by the main theorem. Then for all $Z\in\Lambda^{\emptyset}$

$$H\langle \mathsf{S},\mathsf{S}\rangle \neq H\langle \mathsf{I}, Z\rangle = H\langle Z, \mathsf{I}\rangle.$$

Now define $F \equiv \lambda xy.H\langle x,y\rangle$. Then for all $Z\in\Lambda^{\emptyset}$

$$F\mathsf{S}\,\mathsf{S} \neq F\mathsf{I}Z = FZ\mathsf{I}.$$

This is indeed a counterexample. ∎

2. Proof of the main theorem

In order to prove the main theorem 1.5, let a V-partition determined by S be fixed in this section. By proposition 1.4 it may be assumed that the partition is inhabited.

2.1. Lemma. *Let $\{\mathcal{W}_e\}_{e\in S}$ be an inhabited V-partition.*

(i) *There exists a total recursive function $f = f_S$ such that*

$$\forall e\in S\; W_e = \{f((2e+1)2^n) \mid n\in\mathbb{N}\}.$$

(ii) *There exists a combinator E^S such that*

$$\forall e\in S\; \mathcal{W}_e = \{\mathsf{E}^S \mathbf{c}_{(2e+1)2^n} \mid n\in\mathbb{N}\}.$$

Proof. (i) By elementary recursion theory there exists a recursive function h such that $W_e = \text{Range}(\varphi_{h(e)})$ and $\varphi_{h(e)}$ is total, for all $e\in S$. Observing that e,n are uniquely determined by $k = (2e+1)2^n$, define f by $f(0) = 0$, $f((2e+1)2^n) = \varphi_{h(e)}(n)$.

(ii) Take $\mathsf{E}^S = \mathsf{E}\circ F_S$, where F_S lambda defines f_S and $\mathsf{E}\mathbf{c}_{\#M} = M$ for all $M\in\Lambda^{\emptyset}$. ∎

2.2. Definition. (i) Define

$$\begin{aligned} \mathtt{odd}(0) &= 0;\\ \mathtt{odd}((2e+1)2^n) &= 2e+1. \end{aligned}$$

(ii) Define $M\sim N$ iff $M = N \;\vee\; M = \mathsf{E}_m, N = \mathsf{E}_n$ and $\mathtt{odd}(m) = \mathtt{odd}(n)$, for some m,n.

Notice that $M\sim N$ iff $M = N$ or $\exists e\in S M,N\in\mathcal{W}_e$. Therefore we have to prove that there exists a combinator H such that

$$HM = HN \Leftrightarrow M\sim N.$$

The proof consists in constructing a combinator $H = H^S$ such that

1. $M\sim N \Rightarrow HM = HN$, proposition 2.4;
2. $HM = HN \Rightarrow M\sim N$, proposition 2.9.

The second part of the main theorem easily follows by inspecting the proof.

2.3. DEFINITION. (i) Define

$$\begin{aligned} T &\equiv \lambda xyz.xy(xyz); \\ A &\equiv \lambda fgxyz.fx(a(\mathsf{E}x))[f(\mathsf{S}^+x)y(g(\mathsf{S}^+x))z]; \\ B &\equiv \lambda fgx.f(\mathsf{S}x)(a(\mathsf{E}(Tx))(g(\mathsf{S}^+x))(gx). \end{aligned}$$

(ii) By the double fixed-point theorem there exists terms F, G such that

$$\begin{aligned} F &\twoheadrightarrow AFG; \\ G &\twoheadrightarrow BFG. \end{aligned}$$

To be explicit, write

$$\begin{aligned} D &\equiv (\lambda xy.y(xxy)); \\ Y &\equiv DD; \\ G &\equiv Y(\lambda u.B(Y(\lambda v.Auv))u); \\ F &\equiv Y(\lambda u.AuG). \end{aligned}$$

(iii) Finally define

$$H \equiv \lambda xa.F\mathbf{c}_1(ax)(G\mathbf{c}_1).$$

NOTATION. Write

$$\begin{aligned} F_k &\equiv F\mathbf{c}_k; \\ G_k &\equiv G\mathbf{c}_k; \\ \mathsf{E}_k &\equiv \mathsf{E}\mathbf{c}_k; \\ a_k &\equiv a\mathsf{E}_k; \\ H_k[\ ] &\equiv F_k[\ ]G_k; \\ C_k[\ ] &\equiv F_k a_k([\ ]G_k). \end{aligned}$$

Note that by construction

$$\begin{aligned} F_k MN &\twoheadrightarrow F_k a_k(F_{k+1}MG_{k+1}N); \\ G_k &\twoheadrightarrow F_{k+1}a_{2k}G_{k+1}G_k. \end{aligned}$$

By reducing F, respectively G, it follows that

$$H_k[a_p] \equiv F_k a_p G_k \twoheadrightarrow C_k[H_{k+1}[a_p]] \qquad (1)$$
$$H_k[a_k] \equiv F_k a_k G_k \twoheadrightarrow C_k[H_{k+1}[a_{2k}]] \qquad (2)$$

2.4. PROPOSITION. $M \sim N \Rightarrow HM = HN$.

PROOF. By lemma 2.1 it suffices to show $H\mathsf{E}_k = H\mathsf{E}_{2k}$ for all k.

$$\begin{aligned} H\mathsf{E}_k &= \lambda a.H_1[a_k] \\ &= \lambda a.C_1[C_2[\ldots C_{k-1}[H_k[a_k]]..]], && \text{by (1)}, \\ &= \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a_{2k}]]]..]], && \text{by (2)}, \\ H\mathsf{E}_{2k} &= \lambda a.H_1[a_{2k}] \\ &= \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a_{2k}]]]..]], && \text{by (1)}. \ \blacksquare \end{aligned}$$

As a piece of art we exhibit in more detail the reduction flow (contracted redexes are underlined).

$$
\begin{array}{l}
\underline{H\mathsf{E}_k} \\
\lambda a.\underline{F_1 a_k G_1} \\
\lambda a.F_1 a_1(\underline{F_2 a_2 G_2} G_1) \\
\lambda a.F_1 a_1(F_2 a_2(\underline{F_3 a_k G_3} G_2) G_1) \\
\ldots \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(\ldots(F_k a_k G_k G_{k-1})\ldots) G_2) G_1) \equiv \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(\ldots(F_k a_k \qquad \underline{G_k} \qquad G_{k-1})\ldots) G_2) G_1) \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(\ldots(F_k a_k(F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1})\ldots) G_2) G_1)
\end{array}
$$

And also

$$
\begin{array}{l}
H\mathsf{E}_{2k} \twoheadrightarrow \ldots \twoheadrightarrow \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(\ldots(F_k a_k(F_{k+1} a_{2k} G_{k+1} G_k) G_{k-1})\ldots) G_2) G_1)
\end{array}
$$

For the converse implication we need the fine structure of the reduction.

2.5. Definition. Define

$$
\begin{array}{lcl}
D_k^0[M] & \equiv & F_k(aM) \equiv Y(\lambda u.AuG)\mathbf{c}_k(aM) \\
D_k^1[M] & \equiv & (\lambda y.y(DDy))(\lambda u.AuG)\mathbf{c}_k(aM) \\
D_k^2[M] & \equiv & (\lambda u.AuG)F_k(aM) \\
D_k^3[M] & \equiv & AFG\mathbf{c}_k(aM) \\
D_k^4[M] & \equiv & (\lambda gxyz.F_x(a\mathsf{E}_x)(F_{\mathsf{S}^+x}y(g(\mathsf{S}^+x))z))G\mathbf{c}_k(aM) \\
D_k^5[M] & \equiv & (\lambda xyz.F_x(a\mathsf{E}_x)(F_{\mathsf{S}^+x}yG_{\mathsf{S}^+x}z))\mathbf{c}_k(aM) \\
D_k^6[M] & \equiv & (\lambda yz.F_k(a\mathsf{E}_k)(F_{\mathsf{S}^+\mathbf{c}_k}yG_{\mathsf{S}^+\mathbf{c}_k}z))(aM) \\
D_k^7[M] & \equiv & (\lambda z.F_k(a\mathsf{E}_k)(F_{\mathsf{S}^+\mathbf{c}_k}(aM)G_{\mathsf{S}^+\mathbf{c}_k}z))
\end{array}
$$

2.6. Lemma. *Let $F_k(aM)N$ head-reduce in $8p+q$ steps to W. Then*

$$
\begin{array}{lcll}
W & \equiv & D_k^q[M]N, & \textit{if } p=0; \\
 & \equiv & D_k^q[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^{p-1}(H_{k+1}[M]N)), & \textit{else.}
\end{array}
$$

Proof. Note that $F_k(aM)N \equiv D_k^0[M]N$. Moreover,

$$
\begin{array}{lcll}
D_k^q[M]N & \to_h & D_k^{q+1}[M]N, & \text{for } q<7; \\
D_k^7[M]N & \to_h & D_k^0[\mathsf{E}_k](H_{k+1}[M]N). &
\end{array}
$$

The rest is clear. At steps 16, 24 we obtain for example

$$
\begin{array}{rcl}
D_k^7[\mathsf{E}_k](H_{k+1}[M]N) & \to_h & D_k^0[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])(H_{k+1}[M]G_k)). \\
D_k^7[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])(H_{k+1}[M]G_k)) & \to_h & D_k^0[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^2(H_{k+1}[M]G_k)). \blacksquare
\end{array}
$$

Remember that a standard reduction $\sigma{:}M \twoheadrightarrow_s N$ always consists of a head-reduction followed by an internal reduction:

$$\sigma{:}M \twoheadrightarrow_h W \twoheadrightarrow_i N.$$

NOTATION. Write $M =_{s\leq n} N$ if there are standard reductons of length $\leq n$ from M respectively N to a common reduct Z. Similarly $M =_{i\leq n} N$ for internal standard reductions. Also the notations $=_{s<n}$ and $=_{i<n}$ will be used.

2.7. LEMMA. (i) $D^q_k[M]N =_{i\leq n} D^{q'}_k[M']N' \ \Rightarrow\ q = q'\ \&\ N =_{s\leq n} N'$.
(ii) $D^q_k[M]N =_{i\leq n} D^q_k[M']N'\ \&\ q < 7 \ \Rightarrow\ M =_{s\leq n} M'$.
(iii) $D^7_k[M]N =_{i\leq n} D^7_k[M']N' \ \Rightarrow\ H_{k+1}[M] =_{s\leq n} H_{k+1}[M']$.

PROOF. (i) Suppose $D^q_k[M]N =_{i\leq n} D^{q'}_k[M']N'$. Then By observing where the free variable a occurs one can conclude that $q = q'$. Since the reductions to a common reduct are internal, the positions of N, N' are not changed and hence $N =_{s\leq n} N'$.
(ii) Obvious from the definition of D^q_k.
(iii) In this case it follows that

$$D^0_k[\mathsf{E}_k](H_{k+1}[M]z) =_{i\leq n} D^0_k[\mathsf{E}_k](H_{k+1}[M']z).$$

The conclusion $H_{k+1}[M] =_{s\leq n} H_{k+1}[M']$ depends on the fact that there are the free variables z to mark the residuals. ∎

2.8. LEMMA. *Suppose* $G_k =_{s\leq n} (H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k)$. *Then*

$$H_{k+1}[\mathsf{E}(T\mathbf{c}_k)] =_{s<n} H_{k+1}[M].$$

PROOF. By induction on d. If $d = 0$, then we have $G_k =_{s\leq n} H_{k+1}[M]G_k$. So there are standard reductions of these two terms to a common reduct. Observe that the head-reduction starting with G_k begins as follows.

$$\begin{array}{rcl} G_k & \equiv & Y(\lambda u.B(Y(\lambda v.Avu))u)\mathbf{c}_k \\ & \to_h & (\lambda x.x(Yx))(\lambda u.B(Y(\lambda v.Avu))u)\mathbf{c}_k \\ & \to_h & (\lambda u.B(Y(\lambda v.Avu))u)G\mathbf{c}_k \\ & \to_h & BFG\mathbf{c}_k \\ & \to_h & (\lambda gx.F(\mathsf{S}^+k)(a(\mathsf{E}^S(Tx)))(g(\mathsf{S}^+k))(gx)G\mathbf{c}_k \\ & \to_h & (\lambda x.F(\mathsf{S}^+k)(a(\mathsf{E}^S(Tx)))(G(\mathsf{S}^+k))(Gx))\mathbf{c}_k \\ & \to_h & F(\mathsf{S}^+k)(a(\mathsf{E}^S(T\mathbf{c}_k)))(G(\mathsf{S}^+k))(G\mathbf{c}_k). \end{array}$$

The heads of these terms are not of order 0 except the last one. But $H_{k+1}[X]$ is always of order 0. Therefore the mentioned standard reduction of G_k goes at least to this last term $H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)]G_k$. But then $H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)] =_{s<n} H_{k+1}[M]$.

If $d > 0$, then start the same argument as above, but at the intermediate conclusion

$$H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)]G_k =_{s<n} (H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k),$$

one preceeds by concluding that

$$G_k =_{s<n} H_{k+1}[\mathsf{E}_k]^{d-1}(H_{k+1}[M]G_k)$$

and uses the induction hypothesis. ∎

2.9. PROPOSITION. $H_k[M] = H_k[N] \ \Rightarrow\ M \sim N$.

PROOF. By the standardization theorem it suffices to show for all n that

$$\forall k \in \mathbb{N} [H_k[M] =_{s \leq n} H_k[N] \ \Rightarrow \ M \sim N].$$

This will be done by induction on n. From $H_k[M] =_{s \leq n} H_k[N]$ it follows that

$$\begin{array}{lllll} H_k[M] & \twoheadrightarrow_h & W_M & \twoheadrightarrow_i & Z \\ H_k[N] & \twoheadrightarrow_h & W_N & \twoheadrightarrow_i & Z. \end{array}$$

for some W_M, W_N, Z.

Case 1. W_M, W_N are both reached after < 8 steps. Then by lemma 2.6 $W_M \equiv D_k^q[M]G_k, W_N \equiv D_k^{q'}[N]G_k$. By lemma 2.7(i) it follows that $q = q'$. If $q < 7$, then by 2.7(ii) one has $M = N$ so $M \sim N$. If $q = 7$, then by 2.7(iii) one has $H_{k+1}[M] =_{s<n} H_{k+1}[N]$ and by the induction hypothesis one has $M \sim N$.

Case 2. W_M is reached after $p \geq 8$ steps and W_N after $q < 8$ steps. Then $p = 8d + q$ and, keeping in mind lemma 2.7(i), it follows that $W_M \equiv D_k^q[M]G_k, W_N \equiv D_k^q[E_k]R$, $G_k =_{s<n} R$, where $R \equiv (H_{k+1}[\mathsf{E}_k])^{d-1}(H_{k+1}[N]G_k)$. Then as in case 1 it follows that $M \sim \mathsf{E}_k$. Moreover, by lemma 2.8 $H_{k+1}[\mathsf{E}_{2k}] =_{s<n} H_{k+1}[N]$, so by the induction hypothesis $\mathsf{E}_{2k} \sim N$. So $M \sim \mathsf{E}_k \sim \mathsf{E}_{2k} \sim N$.

Case 3. Both W_M, W_N are reached after ≥ 8 steps. Then

$$\begin{array}{lll} W_M & \equiv & D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k)); \\ W_N & \equiv & D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^{d'}(H_{k+1}[N]G_k)). \end{array}$$

If $d = d'$, then by lemma 2.7

$$(H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k) =_{s<n} (H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[N]G_k),$$

so

$$H_{k+1}[M] =_{s<n} H_{k+1}[N],$$

since $H_{k+1}[X]$ is always of order 0. Therefore by the induction hypothesis $M \sim N$.

If on the other hand, say, $d < d'$, then (writing $d' = d + e$)

$$\begin{array}{lll} W_M & \equiv & D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M] \qquad\qquad G_k \qquad\qquad)); \\ W_N & \equiv & D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[\mathsf{E}_k] \ \underline{((H_{k+1}[\mathsf{E}_k])^{e-1}(H_{k+1}[N]G_k))} \ )). \end{array}$$

so

$$\begin{array}{rll} H_{k+1}[M] & =_{s<n} & H_{k+1}[\mathsf{E}_k] \\ G_k & =_{s<n} & (H_{k+1}[\mathsf{E}_k])^{e-1}(H_{k+1}[N]G_k), \end{array}$$

since $H_{k+1}[X]$ is always of order 0. Therefore by lemma 2.8

$$H_{k+1}[\mathsf{E}_{2k}] \quad =_{s<n} \quad H_{k+1}[N]$$

Therefore by the induction hypothesis twice we obtain $M \sim \mathsf{E}_k \sim \mathsf{E}_{2k} \sim N$. ∎

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