Reflection and its use

from science to meditation

Mathematics

A mathematical phenomenon

Consider

$$1 \qquad 4 \qquad 9 \qquad 16 \qquad 25 \qquad 36 \qquad \cdots$$

What next?

We have the sequence of squares.

Differences

$$\begin{array}{ccccccccccccc} 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & \cdots \\ & 3 & & 5 & & 7 & & 9 & & 11 & & \cdots & \end{array}$$

Differences

$$
\begin{array}{ccccccccccccc}
1 & & 4 & & 9 & & 16 & & 25 & & 36 & & \cdots \\
& 3 & & 5 & & 7 & & 9 & & 11 & & \cdots & \\
& & 2 & & 2 & & 2 & & 2 & & \cdots & &
\end{array}
$$

A theorem

PROPOSITION. *Define*

$$\begin{aligned} a_n &= n^2 \\ b_n &= a_{n+1} - a_n \\ c_n &= b_{n+1} - b_n \end{aligned}$$

Then for all n one has $c_n = 2$.

Visualization:

$$\begin{array}{cccccccccccccccc} n & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & \cdots \\ a_n & 0 & & 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & \cdots \\ b_n & & 1 & & 3 & & 5 & & 7 & & 9 & & 11 & & \cdots & \\ c_n & & & 2 & & 2 & & 2 & & 2 & & 2 & & \cdots & & \end{array}$$

Cubes

Given a sequence $a : a_0, a_1, a_2, \ldots$. Define Da by

$$(Da)_n = a_{n+1} - a_n.$$

PROPOSITION. *Let a^3 be the sequence defined by $a^3_n = n^3$. Then $DDDa^3 = 6$ for all n.*

$$\begin{array}{ccccccccccccc}
0 & & 1 & & 8 & & 27 & & 64 & & 125 & & \cdots \\
& 1 & & 7 & & 19 & & 37 & & 61 & & \cdots & \\
& & 6 & & 12 & & 18 & & 24 & & \cdots & & \\
& & & 6 & & 6 & & 6 & & \cdots & & &
\end{array}$$

In general one has

THEOREM. $D^k a^k = k!$.

The axiomatic-deductive method

Aristotle (384-322 BC)

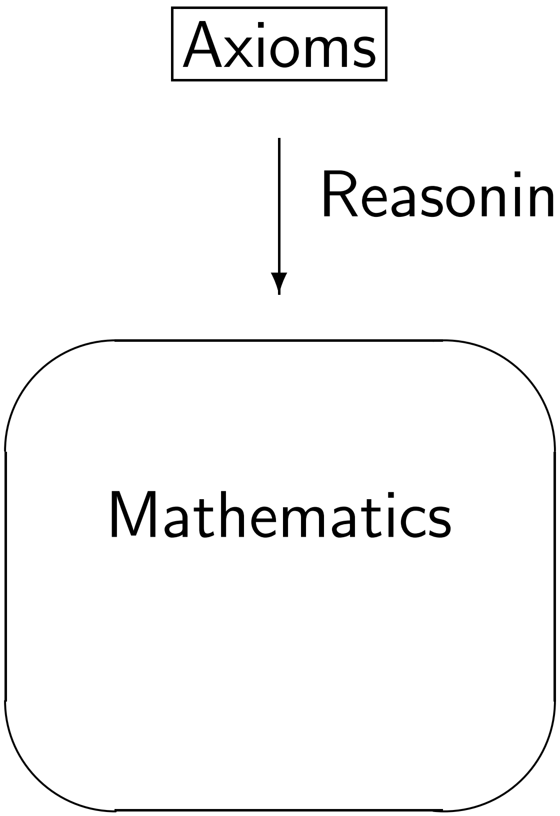
- The axiomatic method

objects	properties
primitive	axioms
defined	derived

- The quest for logic: try to chart reasoning

Aristotle & Phyllis
Aristotle & Phyllis undressed
Aristotle & Phyllis on carpet

Mathematics after Aristotle



Aristotle submissive to Phyllis:
a medieval phantasy.

Peano Axioms for Arithmetic

1. $0 \in \mathbb{N}$
2. $n \in \mathbb{N} \rightarrow Sn \in \mathbb{N}$
3. $Sn = Sm \rightarrow n = m$
4. $\forall n . Sn \neq 0$
5. Let P be a property of natural numbers. Suppose that

   $P(0)$
   $P(n) \rightarrow P(S(n))$ for all natural numbers n.

   Then $P(n)$ for all natural numbers n.

Addition

DEFINITION. Addition can be specified as follows.

$$\begin{aligned} a + 0 &= a \\ a + S(b) &= S(a + b). \end{aligned}$$

PROPOSITION. $\forall a, b, c\ (a + b) + c = a + (b + c)$.

PROOF. Given a, b we have to show $\forall c\ P(c)$, where $P(c) := (a + b) + c = a + (b + c)$. We do this by mathematical induction.

Case $c = 0$. Then $P(c)$ states $(a + b) + 0 = a + (b + 0)$. This holds:

$$\begin{aligned} (a + b) + 0 &= a + b \\ &= a + (b + 0) \end{aligned}$$

Induction step. Suppose $P(c)$ holds, i.e. $(a + b) + c = a + (b + c)$. We call this the *induction hypothesis*. We must show $P(S(c))$ i.e. $(a+b)+S(c) = a+(b+S(c))$. Indeed,

$$\begin{aligned} (a + b) + S(c) &= S((a + b) + c) \\ &= S(a + (b + c)), && \text{by the induction hypothesis,} \\ &= a + S(b + c) \\ &= a + (b + S(c)). \ \blacksquare \end{aligned}$$

The language of Peano arithmetic

Define the context-free abstract grammar.

We need the syntactical categories of *variables*, *terms* and *formulas*.

$$
\begin{aligned}
var &:= x \mid var' \\
term &:= var \mid 0 \mid S\, term \mid term + term \mid term \cdot term \\
form &:= term = term \mid \neg form \mid from \vee form \mid form \,\&\, from \mid \\
&\quad\; form \to form \mid \forall\, var\ form \mid \exists\, var form
\end{aligned}
$$

EXAMPLES.

Variables: x, x', x''

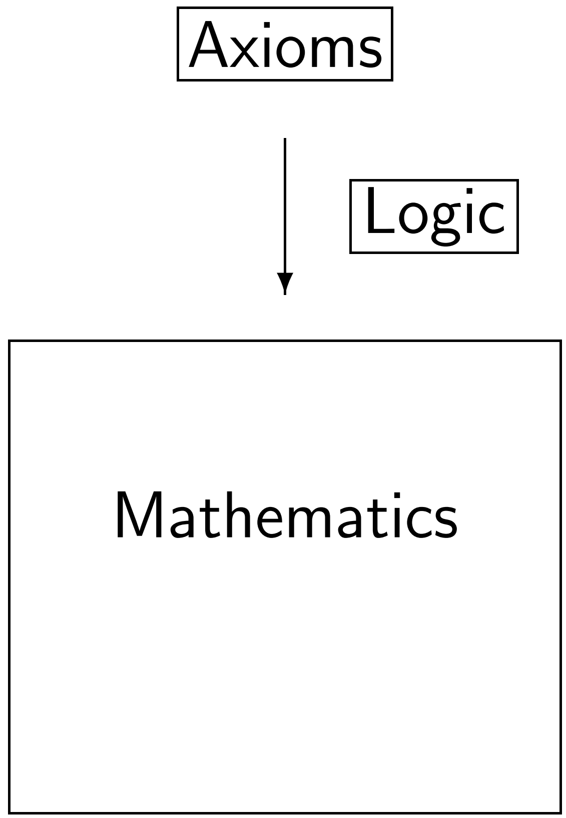
Terms: $x.x + (S0)$, $x.x' + x''$

Formulas: $\forall x \exists x' (x = x' + x')$, $\forall x \forall x' (x.x = x'.x' \to x = x')$

Predicate Logic

	Introduction Rules	Elimination Rules
$\rightarrow$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash (A \rightarrow B)}$	$\dfrac{\Gamma \vdash (A \rightarrow B) \quad \Gamma \vdash A}{\Gamma \vdash B}$
$\&$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash (A \,\&\, B)}$	$\dfrac{\Gamma \vdash (A \,\&\, B)}{\Gamma \vdash A}$ $\dfrac{\Gamma \vdash (A \,\&\, B)}{\Gamma \vdash B}$
$\vee$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash (A \vee B)}$ $\dfrac{\Gamma \vdash B}{\Gamma \vdash (A \vee B)}$	$\dfrac{\Gamma \vdash (A \vee B) \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{C}$
$\forall$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}$ $x \notin \Gamma$	$\dfrac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x := t]}$ t is free in A
$\exists$	$\dfrac{\Gamma \vdash A[x := t]}{\Gamma \vdash \exists x.A}$	$\dfrac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B}{\Gamma \vdash B}$ $x \notin B$
	Start Rule / Absurdum Rule	Classical Negation
	$\dfrac{A \in \Gamma}{\Gamma \vdash A}$ $\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}$	$\dfrac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$ $\neg A := (A \rightarrow \bot)$

Mathematics after Frege



Gödel's theorem

1. Arithmetical statements speak about numbers.
2. (Pythagoras) Everything is a number (after coding).
3. Arithmetical statements speak about everything you want (via coding).
4. Arithmetical statements speak about (other) arithmetical statements.
5. Some arithmetical statements speak about themselves (!).
6. L: This statement is false.
7. G: This statement is unprovable from the Peano axioms.
8. If PA is consistent (free from contradictions), then G is not provable and hence true!

Conclusion: Arithmetic Provability $\neq$ Arithmetic Truth

Coding

Coding $\Sigma_{\text{Peano}} = \{0, S, =, x, ', \neg, \ldots\}$: $\#(0) = 0, \#(S) = 1, \#(=) = 2, \ldots$

$$\begin{aligned} \neg 0 = S0 &\longmapsto \langle \#(\neg), \#(0), \#(=), \#(S), \#(0) \rangle \\ &\longmapsto \langle 5, 0, 2, 1, 0 \rangle \\ &\longmapsto 2^5 3^0 5^2 7^1 11^0 \;=\; 32.1.25.7.1 \;=\; 5600 \\ &= \#(\neg 0 = S0). \end{aligned}$$

From numbers to terms (numerals) $n \longmapsto \underline{n}$.

$$\begin{array}{ccccc} 0 & \longmapsto & \underline{0} & = & 0 \\ 1 & \longmapsto & \underline{1} & = & S0 \\ 2 & \longmapsto & \underline{2} & = & SS0 \\ 3 & \longmapsto & \underline{3} & = & SSS0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

Coding formulas: $\ulcorner A \urcorner = \underline{\#(A)}$.

Examples

One may construct a formula $P_0(x)$ such that

$P_0(\ulcorner A \urcorner)$ states that A starts with an S.

$$P_0(x) = (\exists y\ (y + y = x))\ \&\ \neg(\exists y\ ((y + y) + (y + y)) = x)$$

Similarly one may construct a formula $\mathtt{Prov}(x)$ such that

$\mathtt{Prov}(\ulcorner A \urcorner)$ states that A is provable in PA.
For this it is important that logic can be captured in finitely many rules.

Self-reflection

One can construct a function s_x such that inside PA

$$s_x(\ulcorner A \urcorner, \underline{n}) = \ulcorner A[x{:=}\underline{n}] \urcorner.$$

Here $A[x{:=}t]$ denotes substitution of t for x in A.
Define $d_x(n) = s_x(n, n)$. Then

$$d_x(\ulcorner A \urcorner) = s_x(\ulcorner A \urcorner, \underline{\#(A)}) = \ulcorner A[x{:=}\underline{\#A}] \urcorner = \ulcorner A[x{:=}\ulcorner A \urcorner] \urcorner.$$

Wanted: a formula "Self" stating that it, i.e. "Self", is provable.
Take

$$\begin{array}{rcl} A(x) & = & \texttt{Prov}(d_x(x)) \\ \texttt{Self} & = & A[x{:=}\ulcorner A \urcorner]. \end{array}$$

Indeed,

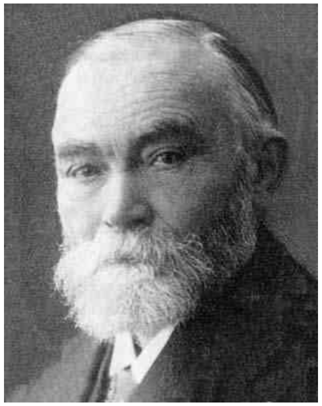
$$\begin{array}{rcll} \texttt{Self} & \leftrightarrow & A[x{:=}\ulcorner A \urcorner], & \text{by definition of } \texttt{Self}, \\ & \leftrightarrow & \texttt{Prov}(d_x(\ulcorner A \urcorner)), & \text{by definition of } A(x), \\ & \leftrightarrow & \texttt{Prov}(\ulcorner A[x{:=}\ulcorner A \urcorner] \urcorner), & \text{by the property of } d, \\ & \leftrightarrow & \texttt{Prov}(\ulcorner \texttt{Self} \urcorner), & \text{by definition of } \texttt{Self}. \end{array}$$

Gödel sentence

Similarly we can construct G such that

$$G : \neg \texttt{Prov}(\ulcorner G \urcorner)$$

Picture Gallery



Frege

Peano

Hilbert

Gödel