21. (i) Definition

- $CL(\mathsf{I},\mathsf{K},\mathsf{S}) = x|\mathsf{I}|\mathsf{K}|\mathsf{S}|PQ$.
- For $P \in CL(\mathsf{I},\mathsf{K},\mathsf{S})$ we define $\lambda^* x.P \in CL(\mathsf{I},\mathsf{K},\mathsf{S})$ by induction on the structure of P as follows

$$\begin{array}{rcl} \lambda^* x.x & \equiv & \mathsf{I}. \\ \lambda^* x.P & \equiv & \mathsf{K}P \text{ if } x \notin FV(P). \\ \lambda^* x.PQ & \equiv & \mathsf{S}(\lambda^* x.P)(\lambda^* x.Q) \text{ otherwise.} \end{array}$$

- We define the map $(-)^* : \Lambda \to CL(\mathsf{I},\mathsf{K},\mathsf{S})$ by

$$\begin{array}{rcl} x^* & \equiv & x. \\ (MN)^* & \equiv & M^* N^*. \\ (\lambda x.M)^* & \equiv & \lambda^* x.M^*. \end{array}$$

(ii) Determine $\mathsf{I}^*, \mathsf{K}^*$ and S^*.
Prove

(iii) $\lambda^* x.P$ is indeed defined for each $P \in CL(\mathsf{I},\mathsf{K},\mathsf{S})$.

(iv) $FV(\lambda^* x.P) = FV(P) - \{x\}$ for $P \in CL(\mathsf{I},\mathsf{K},\mathsf{S})$.

(v) $FV(M^*) = FV(M)$ for $M \in \Lambda$.

(vi) M closed $\Rightarrow$ M^* is a pure product of I, K and S.

(vii) $\lambda^* x.P \twoheadrightarrow_\beta \lambda x.P$ for $P \in CL(\mathsf{I},\mathsf{K},\mathsf{S})$.

(viii) $M^* \twoheadrightarrow_\beta M$ for $M \in \Lambda$.

22. Verify that $\mathsf{SK} \twoheadrightarrow_\beta \lambda xy.y$ and find types A and B such that $\vdash \mathsf{SK} : A$ and $\vdash \lambda xy.y : B$ but $\nvdash \mathsf{SK} : B$.

23. Prove that the rules (Cut) and $(\leq -L)$ are admissible in $\lambda_{\cap\top}^{BCD}$ i.e. prove

$$\begin{array}{rcl} \Gamma, x : B \vdash M : A \;\&\; \Gamma \vdash N : B & \Rightarrow & \Gamma \vdash (M[x := N]) : A. \\ \Gamma, x : B \vdash M : A \;\&\; C \leq B & \Rightarrow & \Gamma, x : C \vdash M : A. \end{array}$$

24. Prove for system $\lambda_{\cap\top}^{BCD}$.
Let $M \in \Lambda^\emptyset$.

$$M \text{ has a normal form } \Rightarrow \exists A[\top \notin A \;\&\; \vdash M : A]$$

You may use that (β-red) and (β-exp) are admissible.