Intersection Types

Henk Barendregt

Nijmegen University

From "Typed lambda claculus" with Dekkers and Statman co-authors

Part III: "Intersection types" with Dezani, Honsell, Alessi co-authors

(Un)typed lambda calculus

Untyped terms

$$\begin{array}{lll}\texttt{term} & ::= & \texttt{var} \mid \texttt{ter ter} \mid \lambda\texttt{var ter} \\ \texttt{var} & ::= & \texttt{x} \mid \texttt{var}' \end{array}$$

Types

$$\begin{array}{lll}\texttt{type} & ::= & \texttt{atom} \mid \texttt{type}{\rightarrow}\texttt{type} \\ \texttt{atom} & ::= & \alpha \mid \texttt{atom}' \end{array}$$

Type-assignment to terms $\lambda\rightarrow$ (Curry [1934])

$$\frac{x{:}A{\in}\Gamma}{\Gamma \vdash x : A}$$

$$\frac{\Gamma \vdash M : A{\rightarrow}B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A{\rightarrow}B}$$

Recursive types $\lambda_=$

The $\lambda_\rightarrow$ types are freely generated from the atoms

The recursive types $\lambda_=$ equate certain of these types

The equation $A = A \rightarrow B$ has as consequence

$\vdash \lambda x.xx : A$

$\vdash (\lambda x.xx)(\lambda x.xx) : B$

There are many ways to make identifications $\longmapsto$ *type algebras*

$$\mathcal{T} = \langle T, \rightarrow \rangle$$

Type-assignment to terms $\lambda\cap$

$$\frac{x{:}A\in\Gamma}{\Gamma\vdash x : A}$$

$$\frac{\Gamma\vdash M : A{\to}B \quad \Gamma\vdash N : A}{\Gamma\vdash MN : B} \qquad \frac{\Gamma, x{:}A\vdash M : B}{\Gamma\vdash \lambda x.M : A{\to}B}$$

$$\frac{\Gamma\vdash M : A_1\cap A_2}{\Gamma\vdash M : A_i} \qquad \frac{\Gamma\vdash M : A_1 \quad \Gamma\vdash M : A_2}{\Gamma\vdash M : A_1\cap A_2}$$

$$\frac{\Gamma\vdash M : A \quad A\leq B}{\Gamma\vdash M : B}$$

$$\frac{}{\Gamma\vdash M : \Omega}$$

Intersection Type Structures

Now we work with *intersection type structures*

$$\mathcal{T} = \langle T, \rightarrow, \leq, \cap, \Omega \rangle$$

$\vdash \lambda x.xx : (A \cap (A \rightarrow B)) \rightarrow B$

$\vdash (\lambda x.xx)(\lambda x.xx) : \Omega$

Subject Reduction

In $\lambda\!\rightarrow$ one has

$$\left.\begin{array}{l}\Gamma \vdash M : A \\ M \twoheadrightarrow_\beta N\end{array}\right\} \;\Rightarrow\; \Gamma \vdash N : A$$

This also holds for $\lambda\cap$, for many intersection type structures $\mathcal{T}$

The converse, *subject expansion*, does not hold for $\lambda\!\rightarrow$

$$\left.\begin{array}{l}\Gamma \vdash N : A \\ M \twoheadrightarrow_\beta N\end{array}\right\} \;\not\Rightarrow\; \Gamma \vdash M : A$$

$\vdash \lambda xy.y : A\to B\to B$ and $\mathsf{SK} \twoheadrightarrow_\beta \lambda xy.y$

but $\not\vdash \mathsf{SK} : A\to B\to B$

In fact one 'only' has

$$\vdash \mathsf{SK} : (B\to C)\to B\to B$$

Subject expansion for $\lambda\cap$

Suppose

$$\vdash P[x := Q] : A$$

where $P \equiv \ldots x \ldots x \ldots x \ldots$

so $\ldots Q \ldots Q \ldots Q \ldots : A$

Each of these occurrences of Q may need another type B_1, B_2, B_3

But then we can give $\lambda x.P$ the type $B_1 \cap B_2 \cap B_3 \to A$

Hence the β-expansion $(\lambda x.P)Q$ also the type A

If the number of occurrences of x in P is 0,

then we may give to $\lambda x.P$ the type $\Omega \to A$

which is consistent as the *empty* intersection

again

$$\vdash (\lambda x.P)Q : A$$

Undecidablity of inhabitation Urzyczyn [1994]

For several $\mathcal{T}$ one has

$$\exists M \in \Lambda^{\emptyset} \vdash^{\mathcal{T}} M : A \text{ is undecidable,}$$

as a predicate in A.

Special Intersection Type Structures

Let $\mathcal{T} = \langle T, \to, \leq, \cap, \Omega \rangle$ be an intersection type structure

$\mathcal{T}$ is *natural* iff

$A \leq \Omega$	(Ω)
$\Omega \leq (\Omega \to \Omega)$	$(\Omega\eta)$
$(A \to B) \cap (A \to C) \leq A \to B \cap C$	$(\to\cap)$
$A' \leq A \ \& \ B \leq B' \ \Rightarrow \ (A \to B) \leq (A' \to B')$	(η)

$\mathcal{T}$ is β-*sound* iff

for all $k \geq 1$ and all $A_1, \ldots, A_k, B_1, \ldots, B_k, C, D \in \mathcal{T}$ one has

$$(A_1 \to B_1) \cap \ldots \cap (A_k \to B_k) \leq (C \to D) \ \Rightarrow \ C \leq A_{i_1} \cap \ldots \cap A_{i_p} \ \& \ B_{i_1} \cap \ldots \cap B_{i_p} \leq D,$$
$$\text{for some } p \geq 0 \text{ and } 1 \leq i_1, \ldots, i_p \leq k$$

β-soundness of $\mathcal{T}$ implies that subject reduction holds in $\lambda\cap^{\mathcal{T}}$
(Coppo, Dezani, Honsell, Longo [1984])

Therefore

$$\left.\begin{array}{l}\Gamma \vdash M : A \\ M =_\beta N\end{array}\right\} \quad \Rightarrow \quad \Gamma \vdash N : A$$

so (for closed M)

$$X_M = \{A \mid \ \vdash M : A\}$$

looks like a λ-model. Indeed, such a set is a *filter* of types. $\left| \begin{array}{l} A, B \in X \ \Rightarrow \ (A \cap B) \in X \\ B \geq A \in X \ \Rightarrow \ B \in X \end{array}\right.$

For filters X, Y one can define application

$$XY = \{B \mid \ \exists A \in Y \ (A \to B) \in X\}$$

is well defined and one has (for many intersection type structures)

$$X_M X_N = X_{MN}$$

Given an intersection type structure $\mathcal{T}$, then $\mathcal{F}^{\mathcal{T}} = \{X \subseteq \mathcal{T} \mid X \text{ is a filter}\}$

is the filter structure over $\mathcal{T}$. If $\mathcal{T}$ is β-sound it is a λ-model.

Extensionality

$\mathcal{F}^{\mathcal{T}}$ is extensional iff

for all $A \in \mathcal{T}$ there are $\vec{B}, \vec{C}, \vec{D}, \vec{E}$, with $\vec{C} = C_1, \ldots, C_k,\ k > 0$ not the top, and

$$
\begin{array}{l}
(B_1 \rightarrow C_1) \cap \ldots \cap (B_k \rightarrow C_k) \cap (B_{k+1} \rightarrow \Omega) \cap \ldots \cap (B_n \rightarrow \Omega) \quad \leq \quad A \\
\&\ A \quad \leq \quad (D_{11} \rightarrow E_{11}) \cap \ldots \cap (D_{1m_1} \rightarrow E_{1m_1}) \cap \\
\qquad \qquad \ldots \\
\qquad \qquad (D_{k1} \rightarrow E_{k1}) \cap \ldots \cap (D_{km_k} \rightarrow E_{km_k}) \\
\&\ \ B_i \leq D_{i1} \cap \ldots \cap D_{im_i} \ \&\ E_{i1} \cap \ldots \cap E_{im_i} \leq C_i, \\
\qquad \text{for } 1 \leq i \leq k.
\end{array}
$$

It is enough that every type A one has

$A \sim (B_1 \rightarrow C_1) \cap \ldots \cap (B_k \rightarrow C_k)$ or $A \geq (B_1 \rightarrow \Omega) \cap \ldots \cap (B_n \rightarrow \Omega)$

Filter models

For these models one has

$$\mathcal{F}^{\mathcal{T}} \models M = N \iff \forall A \in \mathcal{T} [\vdash M : A \iff \vdash N : A]$$

Several known models D_∞ can be written as $D_\infty = \mathcal{F}^{\mathcal{T}}$ for some simple $\mathcal{T}$

New models can be constructed in this way, obtaining wanted properties

Meet semi lattices: $\mathbf{MSL}$ and Algebraic lattices: $\mathbf{ALG}$

A meet semi lattice is a structure with top $\mathcal{S} = \langle \mathcal{S}, \leq, \cap, \Omega \rangle$.

An algebraic lattice is a complete lattice $\mathcal{D} = \langle \mathcal{D}, \sqsubseteq, \sqcup, \top \rangle$,

with countably many compact elements

such that every element is the supremum of capacta below it.

The categories $\mathbf{MSL}$ and $\mathbf{ALG}$ are equivalent.

$\mathbf{MSL}$		$\mathbf{ALG}$
$\mathcal{S}$	$\rightarrow$	$\mathcal{F}^{\mathcal{S}}$
$\mathcal{K}(\mathcal{D})$	$\leftarrow$	$\mathcal{D}$

$\mathcal{K}(\mathcal{D}) = \langle \{ d \in \mathcal{D} | d \text{ is compact} \}, \leq \rangle$ with $d \leq e \;\Leftrightarrow\; e \sqsubseteq d$

$\mathcal{F}^{\mathcal{S}} = \langle \{ X \subseteq \mathcal{S} \mid X \text{ is a filter} \}, \subseteq, \cup \rangle$. One has

$$\begin{array}{rcl} \mathcal{D} & \cong & \mathcal{F}^{\mathcal{K}(\mathcal{D})} \\ \mathcal{S} & \cong & \mathcal{K}(\mathcal{F}^{\mathcal{S}}) \end{array}$$

Details

Let $\mathcal{S}, \mathcal{S}'$ be meet semi-lattices with top

A relation $\mu \subseteq \mathcal{S} \times \mathcal{S}'$ is an *approximable mapping* between $\mathcal{S}$ and $\mathcal{S}'$ iff

for all $s, t \in \mathcal{S}$ and $s', t', t'_1, t'_2 \in \mathcal{S}'$

$$
\begin{array}{ll}
(a) & \Omega \; \mu \; \Omega' \\
(b) & t \leq s \; \mu \; s' \leq t' \;\Rightarrow\; t \; \mu \; t' \\
(c) & s \; \mu \; t'_1 \;\&\; s \; \mu \; t'_2 \;\Rightarrow\; s \; \mu \; (t'_1 \cap t'_2)
\end{array}
$$

$\mathcal{M}(\mathcal{S}, \mathcal{S}') = \{\mu \mid \mu$ is an appoximable mapping between $\mathcal{S}$ and $\mathcal{S}'\}$

This makes $\mathbf{MSL}$ into a category

On $\mathbf{ALG}$ one considers Scott continuous maps as morphisms

Both categories are being strengthened: An **MSL** $\mathcal{S} = \langle \mathcal{S}, \leq, \cap, \Omega \rangle$ enriched with an arrow, becomes a intersection type structure

$$\mathcal{S} = \langle \mathcal{S}, \leadsto, \leq, \cap, \Omega \rangle$$

If we require naturality we obtain the category **NTS**.

A **NLS** is an $\mathcal{D} \in \mathbf{ALG}$ enriched with operators

$$\begin{array}{ll} F: & \mathcal{D} \to \mathcal{D} \to \mathcal{D} \\ G: & [\mathcal{D} \to \mathcal{D}] \to \mathcal{D} \end{array}$$

such that $\begin{array}{lll} F \circ G & \sqsupseteq & 1_{\mathcal{D} \to \mathcal{D}} \\ G \circ F & \sqsubseteq & 1_{\mathcal{D}} \end{array}$

As before, **NTS** and **NLS** are equivalent: $\mathcal{D} \cong \mathcal{F}^{\mathcal{K}(\mathcal{D})}$ and $\mathcal{S} \cong \mathcal{K}(\mathcal{F}^{\mathcal{S}})$, where

$\mathcal{K}(\mathcal{D}) = \langle \{d | d \text{ is compact}\}, \leadsto, \leq \rangle$ with $d \leadsto e = G(d \Rightarrow e)$
$\mathcal{F}^{\mathcal{S}} = \langle \{X \subseteq \mathcal{S} \mid X \text{ is a filter}\}, \subseteq, \cup, F, G \rangle$, with

$$\begin{array}{rcl} F(X)(Y) & = & XY, \\ G(f) & = & \uparrow\{a \to b \in \mathcal{S} \mid b \in f(\uparrow a)\}. \end{array}$$

Classical models

D_∞ depends on D_0 and the pair $i_0 : D_0 \to D_0 \to D_0, j_0 : [D_0 \to D_0]$.

Scott took

$$D_0 = \{0 \sqsubseteq 1\} \quad i_0(d)(e) = d \quad j_0(f) = f(0)$$

For the resulting D_∞ one has

$$D_\infty = \mathcal{F}^{\text{Scott}}$$

with type structure Scott obtained by atoms $\{1 \leq 0 = \Omega\}$ with $0 \leadsto 1 = 1$.

Park took

$$D_0 = \{0 \sqsubseteq 1\} \quad i_0(d)(e) = (1 \Rightarrow d)(e) \quad j_0(f) = f(1)$$

For the resulting D_∞ one has

$$D_\infty = \mathcal{F}^{\text{Park}}$$

with type structure Park obtained by atoms $\{1 \leq 0 = \Omega\}$ with $1 \leadsto 1 = 1$.

New models

Coppo, Dezani and Zacchi [1987]

$\{0 \leq 1 \leq \Omega\}$ with $1 \leadsto 0 = 0, \quad 0 \leadsto 1 = 1$ gives a model $\mathcal{D} = \mathcal{F}^{\mathrm{CDZ}}$ with

$$M \text{ has a nf } \Leftrightarrow [\![M]\!]^{\mathcal{D}} \supseteq \uparrow 1$$

This model $\mathcal{D}$ also can be described in a traditional way

$$D_0 = \{\Omega \sqsubseteq 1 \sqsubseteq 0\}$$

$$i_0(1) = 0 \Rightarrow 1$$
$$i_0(0) = 1 \Rightarrow 0$$

$$j_0(f) = \sqcup\{d \in D_0 \mid i_0(d) \sqsubseteq f\}$$

and one has

$$M \text{ has a nf } \Leftrightarrow [\![M]\!]^{D_\infty} \sqsupseteq 1$$

$\mathcal{F}^{\nabla_M} \models (\lambda x.xx)(\lambda x.xx) = M$ (Fabio Alessi [1991])

Define

$$
\begin{array}{llcl}
1. & \mathbb{C}^{\nabla_0} & = & \{\Omega, \omega\} \\
2. & \nabla_0 & = & (A \to B) \cap (A \to C) \leq (A \to (B \cap C)) \\
 & & & (A \leq \Omega) \\
 & & & \Omega \sim (\Omega \to \Omega) \\
 & & & \Omega \to \omega \sim \omega \\
 & & & \dfrac{A' \leq A \quad B \leq B'}{(A \to B) \leq (A' \to B')} \\
3. & \mathbb{C}^{\nabla_{n+1}} & = & \mathbb{C}^{\nabla_n} \cup \{\xi_{\langle n,m \rangle} \mid m \in \mathbb{N}\} \\
4. & \nabla_{n+1} & = & \nabla_n \cup \{\xi_{\langle n,m \rangle} \sim (\xi_{\langle n,m \rangle} \to W_{\langle n,m \rangle})\}
\end{array}
$$

where $\langle W_{\langle n,m \rangle} \rangle_{m \in \mathbb{N}}$ is any enumeration of the set

$$\{A \mid \vdash^{\nabla_n} M : A\}.$$

Finally set ∇_M as follows:

$$\mathbb{C}^{\nabla_M} = \bigcup_{n \in \mathbb{N}} \mathbb{C}^{\nabla_n}; \quad \nabla_M = \bigcup_{n \in \mathbb{N}} \nabla_n.$$

$\mathbf{ALG}^s$ same objects as $\mathbf{ALG}$ but strict maps as morphisms: $f(\bot) = \bot$.

$\mathbf{NLS}^s$ elements of $\mathbf{ALG}$ extended with $F : [\mathcal{D} \rightarrow_s [\mathcal{D} \rightarrow_s \mathcal{D}]]$, $G : [[\mathcal{D} \rightarrow_s \mathcal{D}] \rightarrow_s \mathcal{D}]$.

$\mathbf{MSL}^s$ consisting of $\mathcal{S} = \langle S, \leq, \cap \rangle$ not necessarily with a top.

$\mathbf{NTS}^s$ elements of $\mathbf{NTS}^s$ extended with $\leadsto$ s.t. it is *restricted natural*

$$(A \rightarrow B) \cap (A \rightarrow C) \leq A \rightarrow B \cap C \qquad (\rightarrow\text{-}\cap)$$

$$A' \leq A \;\&\; B \leq B' \;\Rightarrow\; (A \rightarrow B) \leq (A' \rightarrow B') \qquad (\eta)$$

$$\mathcal{F}_s^{\mathcal{S}} \;=\; \{X \subseteq \mathcal{S} \mid X \text{ is a \textit{strict} filter over } \mathcal{S}\} \qquad \text{(allowing the empty filter)}$$

$$\mathcal{K}^s(\mathcal{D}) \;=\; \mathcal{K}(\mathcal{D})/\bot$$

As before, $\mathbf{NTS}^s$ and $\mathbf{NLS}^s$ are equivalent

$$\mathcal{D} \cong \mathcal{F}_s^{\mathcal{K}_s(\mathcal{D})} \text{ and } \mathcal{S} \cong \mathcal{K}_s(\mathcal{F}_s^{\mathcal{S}})$$

In this way models of the λI-calculus can be obtained.

A proper λI-model (Honsell, Lenisa [1999])

Define the intersection type structure

$$\mathcal{S} = \langle \mathbb{T}(\{\varphi, \omega\}) / \sim, \leq, \cap, \rightarrow \rangle$$

with $\omega \leq \varphi$ and $(\varphi {\rightarrow} \omega) \sim \omega, \ (\omega {\rightarrow} \varphi) \sim \varphi$.

Then

$$\mathcal{F}_s^{\mathcal{S}} \text{ is a } \lambda\text{I-model.}$$

One has

$$\mathrm{Th}(\mathcal{F}_{\mathrm{s}}^{\mathcal{S}}) \text{ is the unique maximal sensible } \lambda\text{I-theory.}$$

It is extensional and equates all terms without nf.