Week 2

In-class problems

1. (a) Let g, h be λ-defined by G, H. Find a $F \in \Lambda^\emptyset$ that is λ-defining

$$\begin{array}{rcl} f(0,y) & = & g(y) \\ f(x+1,y) & = & h(f(x,y),x,y). \end{array}$$

   (b) Lambda define the predecessor (having on 0 the value 0 itself).

2. (a) Show that $\mathsf{times}\ \mathbf{c}_n\ \mathbf{c}_m =_\beta \mathbf{c}_{n.m}$, for all $n, m \in \mathbb{N}$.
   (b) Show that $\mathbf{c}_n \mathbf{c}_m =_\beta \mathbf{c}_{m^n}$, for all $n, m \in \mathbb{N}$.

3. (a) Define the characteristic function $K_\leq$ of the relation $\leq$. Show that it is λ-definable.
   (b) Show that $L(n,m) = \lceil {}^n \log m \rceil$ is λ-definable. Here, for $r \in \mathbb{R}$ the integer $\lceil r \rceil$ is its ceiling, the least element $n \in \mathbb{Z}$ such that $r \leq n$. [Hint. Use 2(b) and (a).]

4. Give an inductive definition of trees with natural numbers at the leaves.

   (a) Specify the function mirror on trees. Show that the function $\mathsf{mirror} \in \Lambda^\emptyset$ (given on the slides) defines it.
   (b) Specify the function that squares all leaves in a tree, leaving the tree structure the same. Construct a λ-defining term for this function.

5. (Klop) Define

$$\begin{array}{rcl} \$ & \triangleq & \lambda abcdefghijklmnopqstuvwxyzr.r(thisisafixedpointcombinator) \\ € & \triangleq & \$. \end{array}$$

   Show that € is a reducing fixed point combinator: $€f \twoheadrightarrow_\beta f(€f)$.

Take-home problems

1. Show that there are no $F_1, F_2 \in \Lambda^{\emptyset}$ such that

$$F_1(xy) = x \ \& \ F_2(xy) = y.$$

[Hint. F_1 alone doesn't even exist.]

2. (Petorossi) Show that C has no nf, where

$$\begin{array}{rcl} \mathsf{A} & \triangleq & \mathsf{SSS} \\ \mathsf{B} & \triangleq & \mathsf{SAA} \\ \mathsf{C} & \triangleq & \mathsf{BB}. \end{array}$$

3. Find a different representation $n \longmapsto \mathbf{d}_n$ of the numbers with $\mathbf{d}_n \in \Lambda^{\emptyset}$ such that $|\mathbf{d}_n| = O(\lg n)$ and there are $F, G \in \Lambda^{\emptyset}$ satisfying

$$F\mathbf{d}_n =_\beta \mathbf{c}_n \ \& \ G\mathbf{c}_n =_\beta \mathbf{d}_n.$$

Show that the terms do what is intended.

4. (a) Show that commutativity or associativity in $\boldsymbol{\lambda}$ does not hold in the following strong sense. Adding to $\boldsymbol{\lambda}$ one of the following axioms makes it inconsistent.

$$xy = yx \quad (1)$$
$$(xy)z = x(yz) \quad (2)$$

(b) Adding either (1) or (2) to CL does not make it inconsistent; show that for this one *needs* an axiom scheme like

$$XY = YX \text{ for all terms } X, Y.$$

5. (Barendregt, Klop, Dezani)

(a) Show that there are $A, B, C, E \in \Lambda^{\emptyset}$ that behave like Klein's four-group with multiplication table below. [Hint. First solve (b).]

	A	B	C	E
A	E	C	B	A
B	C	E	A	B
C	B	A	E	C
E	A	B	C	E

(b) Let $f: \mathbb{N}^2 \to \mathbb{N}$ be computable. Construct $=_\beta$-distinct closed λ-terms $X_0, X_1, X_2 \ldots$ such that for all $n, m \in \mathbb{N}$ one has

$$X_n X_m =_\beta X_{f(n,m)}.$$

[Hint. Try $X_n \triangleq \langle A, \mathbf{c}_n \rangle$.]