Exercises course Lambda Calculus, week 12 (May 14, 2012)

Let $\mathcal{M}_n = \mathcal{M}_{\{1,\ldots,n\}}$ and $c_i = (\lambda fx.f^i x) \in \Lambda^\emptyset(1 \to 0 \to 0)$. Write

$$\mathcal{E}_{\beta\eta}(A) = \{M = N \mid M, N \in \Lambda^\emptyset(A) \;\&\; M =_{\beta\eta} N\}.$$

Define

$$\mathcal{M} \equiv_A \mathcal{N} \iff \forall M, N \in \Lambda^\emptyset(A).[\mathcal{M} \models M = N \iff \mathcal{N} \models M = N].$$

Define

$$\mathrm{Th}(\mathcal{M})(A) = \{M = N \mid M, N \in \Lambda^\emptyset(A) \;\&\; \mathcal{M} \models M = N\}.$$

Let $\mathcal{M}_1, \ldots, \mathcal{M}_5$ be the canonical term models.

1. (a) Show that for $i, j \in \mathbb{N}$ one has

   $$\mathcal{M}_n \models c_i = c_j \iff i = j \;\vee\; [i, j \geq n{-}1 \;\&\; \forall k_{1 \leq k \leq n}.i \equiv j (\mathrm{mod}\; k)].$$

   [Hint. For $a \in \mathcal{M}_n(0), f \in \mathcal{M}_n(1)$ define the trace of a under f as

   $$\{f^i(a) \mid i \in \mathbb{N}\},$$

   directed by $G_f = \{(a, b) \mid f(a) = b\}$, which by the pigeonhole principle is 'lassoo-shaped'. Consider the traces of 1 under the functions f_n, g_m with $1 \leq m \leq n$, where

   $$\begin{array}{rcll} f_n(k) & = & k+1, & \text{if } k < n, \\ & = & n, & \text{if } k = n, \end{array} \quad \text{and} \quad \begin{array}{rcll} g_m(k) & = & k+1, & \text{if } k < m, \\ & = & 1, & \text{if } k = m, \\ & = & k, & \text{else.]} \end{array}$$

   (b) Conclude that $\mathcal{M}_5 \models c_4 = c_{64}$, $\mathcal{M}_6 \not\models c_4 = c_{64}$, and $\mathcal{M}_6 \models c_5 = c_{65}$.

2. Show that $\mathcal{M}_2$ and $\Lambda[\{c^0, d^0\}]$ satisfy different equations.

3. Show that $\mathcal{M}_n \equiv_{\{1\to0\to0\}} \mathcal{M}_m \iff n = m$.

4. Show directly that $\bigcap_n \mathrm{Th}(\mathcal{M}_n)(1) = \mathcal{E}_{\beta\eta}(1)$.

5. Show that $\bigcap_n \mathrm{Th}(\mathcal{M}_n) = \mathrm{Th}(\mathcal{M}_\mathbb{N}) = \mathcal{E}_{\beta\eta}$.

6. Consider the following equations.

   (1) $\lambda f{:}1\lambda x{:}0.fx = \lambda f{:}1\lambda x{:}0.f(fx)$;

   (2) $\lambda f, g{:}1\lambda x{:}0.f(g(g(fx))) = \lambda f, g{:}1\lambda x{:}0.f(g(f(gx)))$;

   (3) $\lambda F{:}3\lambda x{:}0.F(\lambda f_1{:}1.f_1(F(\lambda f_2{:}1.f_2(f_1 x)))) =$
   $\lambda F{:}3\lambda x{:}0.F(\lambda f_1{:}1.f_1(F(\lambda f_2{:}1.f_2(f_2 x))))$.

   (4) $\lambda h{:}1_2\lambda x{:}0.h(hx(hxx))(hxx) = \lambda h{:}1_2\lambda x{:}0.h(hxx)(h(hxx)x)$.

(a) Show that 1 holds in $\mathcal{M}_1$, but not in $\mathcal{M}_2$.

(b) Show that 2 holds in $\mathcal{M}_2$, but not in $\mathcal{M}_3$.

(c) Show that 3 holds in $\mathcal{M}_3$, but not in $\mathcal{M}_4$.

(d) Show that 4 holds in $\mathcal{M}_4$, but not in $\mathcal{M}_5$.

7. Construct six pure closed terms of the same type in order to show that the five canonical theories are maximally different. I.e. we want terms $M_1, \ldots, M_6$ such that in $\mathtt{Th}(\mathcal{M}_5)$ the $M_1, \ldots, M_6$ are mutually different; also $M_6 = M_5$ in $\mathtt{Th}(\mathcal{M}_4)$, but different from $M_1, \ldots, M_4$; also $M_5 = M_4$ in $\mathtt{Th}(\mathcal{M}_3)$, but different from $M_1, \ldots, M_3$; also $M_4 = M_3$ in $\mathtt{Th}(\mathcal{M}_2)$, but different from M_1, M_2; also $M_3 = M_2$ in $\mathtt{Th}(\mathcal{M}_1)$, but different from M_1; finally $M_2 = M_1$ in $\mathtt{Th}(\mathcal{M}_0)$. [Hint. Use the previous exercise and a polynomially defined pairing operator.]

8. Following `<www.cs.ru.nl/ henk/book.pdf>` show that $\mathcal{M}_1$ is decidable. Similarly for $\mathcal{M}_5$.

9. Investigate whether $\mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_4$ are decidable.