Lambda Calculus

Week 12

The canonical term models for $\lambda_{\to}$

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Two version of $\lambda_{\to}$

Curry version (type assignment). $\Lambda^{\Gamma}(A) \triangleq \{M \in \Lambda \mid \Gamma \vdash M : A\}$ with

(axiom)	$\Gamma \vdash x : A$	if $(x{:}A) \in \Gamma$
($\to$-E)	$\dfrac{\Gamma \vdash M : (A \to B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B}$	
($\to$-I)	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A \to B)}$	

Church version (typed terms). $\Lambda^{\mathsf{ch}} \triangleq \bigcup_{A \in \mathbb{T}} \Lambda^{\mathsf{ch}}_{\to}(A)$ with

		$x^A \in \Lambda^{\mathsf{ch}}_{\to}(A)$
$M \in \Lambda^{\mathsf{ch}}_{\to}(A \to B) \quad N \in \Lambda^{\mathsf{ch}}_{\to}(A)$	$\Rightarrow$	$(MN) \in \Lambda^{\mathsf{ch}}_{\to}(B)$
$M \in \Lambda^{\mathsf{ch}}_{\to}(B)$	$\Rightarrow$	$(\lambda x^A.M) \in \Lambda^{\mathsf{ch}}_{\to}(A \to B)$

Note $\vdash \lambda x.x : 0 \to 0$, $\quad \vdash \lambda x.x : (0 \to 0) \to (0 \to 0)$, but $(\lambda x^A.x) \in \Lambda^{\mathsf{ch}}_{\to}(A \to A)$.

Canonical translations between $\lambda{\to}^{\mathrm{CH}}$ and $\lambda{\to}^{\mathrm{CU}}$

Def. There is a *forgetful map* $|\cdot| : \Lambda^{\mathsf{ch}} \to \Lambda$

$$
\begin{aligned}
|x^A| &\triangleq x \\
|MN| &\triangleq |M||N| \\
|\lambda x{:}A.M| &\triangleq \lambda x.|M|
\end{aligned}
$$

Def. Let $M{\in}\Lambda^{\mathsf{ch}}$. Write

$$\Gamma_M \triangleq \{x{:}A \mid x^A{\in}\mathrm{FV}(M)\}$$

Prop. (i) Let $M{\in}\Lambda^{\mathsf{ch}}$. Then

$$M{\in}\Lambda^{\mathsf{ch}}_{\to}(A) \;\Rightarrow\; \Gamma_M \vdash |M| : A$$

(ii) Let $M{\in}\Lambda$. Then

$$\Gamma \vdash M : A \;\Leftrightarrow\; \exists M'{\in}\Lambda^{\mathsf{ch}}_{\to}(A).|M'| \equiv M$$

Type structures

Let $\mathbb{T} = \mathbb{T}^{\{0\}}$ and $\lambda_{\to} = \lambda_{\to}^{\mathsf{ch}}$

Def. Let $\mathcal{M} = \{\mathcal{M}(A)\}_{A \in \mathbb{T}}$ with $\mathcal{M}(A) \neq \emptyset$, for $A \in \mathbb{T}$.

(i) $\mathcal{M}$ is called a type structure for $\lambda_{\to}$ if

$$\mathcal{M}(A \to B) \subseteq \mathcal{M}(B)^{\mathcal{M}(A)} \triangleq \{f \mid f : \mathcal{M}(A) \to \mathcal{M}(B)\}$$

(ii) Let $\mathcal{M}$ be provided with *application operators*

$$\begin{aligned} (\mathcal{M}, \cdot) &= (\{\mathcal{M}(A)\}_{A \in \mathbb{T}}, \{\cdot_{A,B}\}_{A,B \in \mathbb{T}}) \\ \cdot_{A,B} &: \mathcal{M}(A \to B) \times \mathcal{M}(A) \to \mathcal{M}(B). \end{aligned}$$

(iii) A *typed applicative structure* is such an $(\mathcal{M}, \cdot)$ satisfying *extensionality*:

$$\forall f, g \in \mathcal{M}(A \to B)\, [[\forall a \in \mathcal{M}(A)\ f \cdot_{A,B} a = g \cdot_{A,B} a] \ \Rightarrow\ f = g].$$

Prop. The notions of type structure and typed applicative structure are equivalent

Def. $\mathcal{M}$ is called *trivial* if $\mathcal{M}(0)$ is a singleton. Then $\mathcal{M}(A)$ is a singleton for all $A \in \mathbb{T}$

Full type structures

Let X be a set. The *full type structure* $\mathcal{M} = \mathcal{M}_X$ over X is

$$\begin{aligned} \mathcal{M}(0) &\triangleq X \\ \mathcal{M}(A{\to}B) &\triangleq \mathcal{M}(B)^{\mathcal{M}(A)}, \qquad \text{for all } A, B{\in}\mathbb{T}. \end{aligned}$$

Def. $\mathcal{M}_n \triangleq \mathcal{M}_{\{0,\ldots,n-1\}}$

Prop. $\mathcal{M}_2 \models \mathsf{c}_1 = \mathsf{c}_3 \neq \mathsf{c}_2$

Proof. Let $1 = 0{\to}0$. Then

$$\mathcal{M}_2(1) = \{\mathsf{I}_0, \mathsf{K}0, \mathsf{K}1, \mathtt{ff}\}$$

with $\mathtt{ff}(i) = 1 - i$. Note that $\forall f{\in}\mathcal{M}_2(1).f \circ f \circ f = f$. Therefore $\mathcal{M}_2 \models \mathsf{c}_3 = \mathsf{c}_1$.

Note also that $\mathtt{ff} \circ \mathtt{ff} = \mathsf{I} \neq \mathtt{ff}$. Hence $\mathcal{M}_2 \models \mathsf{c}_2 \neq \mathsf{c}_1$. ∎

In the exercises you will show that

$$\mathcal{M}_5 \models \mathsf{c}_4 = \mathsf{c}_{64}, \; \mathcal{M}_6 \models \mathsf{c}_4 \neq \mathsf{c}_{64}, \; \mathcal{M}_6 \models \mathsf{c}_5 = \mathsf{c}_{65}$$

In fact that for $i, j{\in}\mathbb{N}$ one has

$$\mathcal{M}_n \models \mathsf{c}_i = \mathsf{c}_j \;\Leftrightarrow\; i = j \;\vee\; [i, j \geq n{-}1 \;\&\; \forall k_{1\leq k\leq n}.i \equiv j (\mathrm{mod}\ k)]$$

Term models

Def. Let $\mathcal{D}$ be a set of typed variables seen as constants. (i) Write

$$\Lambda^{\mathcal{D}}(A) \triangleq \{M \in \Lambda(A) \mid \mathrm{FV}(M) \subseteq \mathcal{D}\}$$

(ii) $\mathcal{D}$ is called *sufficient* if for every $A \in \mathbb{T}_0$ there is a closed term $M \in \Lambda^{\mathcal{D}}(A)$

For example $\{x^0\}, \{F^2, f^1\}$ are sufficient. But $\{f^1\}, \{\Psi^3, f^1\}$ are not. Note that

$$\mathcal{D} \text{ is sufficient } \Leftrightarrow \Lambda^{\mathcal{D}}(0) \neq \emptyset$$

Def. Let $M, N \in \Lambda^{\mathcal{D}}(A)$ with $A = A_1 \to \ldots \to A_a \to 0$.

(i) M is $\mathcal{D}$-*extensionally equivalent* with N, notation $M \approx^{\mathrm{ext}}_{\mathcal{D}} N$ if

$$\forall t_1 \in \Lambda^{\mathcal{D}}(A_1) \ldots t_a \in \Lambda^{\mathcal{D}}(A_a). M\vec{t} =_{\beta\eta} N\vec{t}.$$

If $a = 0$, then $M, N \in \Lambda^{\mathcal{D}}(0)$; in this case $M \approx^{\mathrm{ext}}_{\mathcal{D}} N \Leftrightarrow M =_{\beta\eta} N$.

(ii) M is $\mathcal{D}$-*observationally equivalent* with N, notation $M \approx^{\mathrm{obs}}_{\mathcal{D}} N$ if

$$\forall F \in \Lambda^{\mathcal{D}}(A \to 0)\, FM =_{\beta\eta} FN$$

Theorem! For all $M, N \in \Lambda^{\mathcal{D}}(A)$ one has

$$M \approx^{\mathrm{ext}}_{\mathcal{D}} N \Leftrightarrow M \approx^{\mathrm{obs}}_{\mathcal{D}} N.$$

We therefore can write $M \approx_{\mathcal{D}} N$.

Term models

Def. Let $\mathcal{D}$ be sufficient. Define

$$\mathcal{M}[\mathcal{D}] \triangleq \Lambda^{\mathcal{D}}/{\approx_{\mathcal{D}}},$$

with application defined by

$$[F]_{\mathcal{D}}[M]_{\mathcal{D}} \triangleq [FM]_{\mathcal{D}}.$$

Here $[-]_{\mathcal{D}}$ denotes an equivalence class modulo $\approx_{\mathcal{D}}$

Prop. $\mathcal{M}[\mathcal{D}]$ is an extensional typed applicative structure satisfying

$$\forall M, N \in \Lambda^{\mathsf{ch}}. [\![M]\!]^{\mathcal{M}[\mathcal{D}]} = [M]_{\approx_{\mathcal{D}}} \tag{1}$$

Proof. That application is well defined follows from

$$F \approx_{\mathcal{D}} F' \ \& \ M \approx_{\mathcal{D}} M' \ \Rightarrow \ FM \approx_{\mathcal{D}} F'M \approx_{\mathcal{D}} F'M'$$

For the first equality we need $\approx^{\text{ext}}_{\mathcal{D}}$, for the second $\approx^{\text{obs}}_{\mathcal{D}}$. By induction on open terms $M \in \Lambda^{\mathcal{D}}$ it follows that for (require $\forall d \in \mathcal{D}. \rho(d) = [d]_{\mathcal{D}}$)

$$[\![M]\!]_{\rho} = [M[\vec{x}:=\rho(x_1), \ldots, \rho(x_n)]]_{\mathcal{D}}.$$

Therefore (1). Extensionality follows as $\approx_{\mathcal{D}}$ is $\approx^{\text{ext}}_{\mathcal{D}}$. ∎

The five canonical term models

These are

$$\begin{array}{lcll} \mathcal{M}_0 & \triangleq & \mathcal{M}[\{\mathsf{c}^0\}] & \text{the 'trivial model'} \\ \mathcal{M}_1 & \triangleq & \mathcal{M}[\{\mathsf{c}^0, d^0\}] & \text{the 'minimal model'} \\ \mathcal{M}_2 & \triangleq & \mathcal{M}[\{\mathsf{c}^0, f^1\}] & \\ \mathcal{M}_3 & \triangleq & \mathcal{M}[\{f^1, g^1, \mathsf{c}^0\}] & \\ \mathcal{M}_4 & \triangleq & \mathcal{M}[\{\Phi^3, \mathsf{c}^0\}] & \\ \mathcal{M}_5 & \triangleq & \mathcal{M}[\{b^{1_2 \to 0 \to 0}, \mathsf{c}^0\}] & \text{the 'maximal model'} \end{array}$$

Prop! Let M, N have the same type.

(i) $\mathcal{M}_0 \models \mathsf{c}_0 = \mathsf{c}_1$, hence $\mathcal{M}_0 \models M = N$. Therefore $\mathcal{M}_0$ is the 'trivial model'.

(ii) $\mathcal{M}_1 \models M = N \;\Leftrightarrow\; \lambda + M = N \nvdash \mathsf{c}_0 = \mathsf{c}_1$.

(iii) $\mathcal{M}_5 \models M = N \;\Leftrightarrow\; M =_{\beta\eta} N$

Reducibility of types

Def. Let $A, B \in \mathbb{T}_0$

(i) We say that A is $\beta\eta$-reducible, notation $A \leq_{\beta\eta} B$, if for some closed term $\Phi{:}A \to B$ one has for all closed $M_1, M_2{:}A$

$$M_1 =_{\beta\eta} M_2 \;\Leftrightarrow\; \Phi M_1 =_{\beta\eta} \Phi M_2,$$

i.e. equalities between terms of type A can be uniformly translated to those of type B.

(ii) Write $A \sim_{\beta\eta} B$ iff $A \leq_{\beta\eta} B$ & $B \leq_{\beta\eta} A$.

(iii) Write $A <_{\beta\eta} B$ for $A \leq_{\beta\eta} B$ & $B \not\leq_{\beta\eta} A$.

Lemma. $A = A_1 \to \dots \to A_a \to 0$ and $B = A_{\pi(1)} \to \dots \to A_{\pi(a)} \to 0$, where π is a permutation of the set $\{1, \dots, a\}$. Then

(i) $B \leq_{\beta\eta} A$

(ii) $A \sim_{\beta\eta} B$.

Proof. (i) We have $B \leq_{\beta\eta} A$ via

$$\Phi \equiv \lambda m{:}B \lambda x_1 \dots x_a . m x_{\pi(1)} \dots x_{\pi(a)}.$$

(ii) By (i) applied to π^{-1}. ∎

Theorem!

0	$<_{\beta\eta}$	
$0\to 0$	$<_{\beta\eta}$	
$0\to 0\to 0$	$<_{\beta\eta}$	
$\dots$	$<_{\beta\eta}$	
$0^k\to 0$	$<_{\beta\eta}$	with $k > 2$
$\dots$	$<_{\beta\eta}$	
$1\to 0\to 0$	$<_{\beta\eta}$	
$1\to 1\to 0\to 0$	$<_{\beta\eta}$	
$3\to 0\to 0$	$<_{\beta\eta}$	
$1_2\to 0\to 0$		where $1_2 \triangleq 0^2\to 0$

Moreover, every type A is equivalent ($\sim_{\beta\eta}$) to one of these

Remember ($0 <_{\beta\eta} 1 <_{\beta\eta}$)

$$1_2 <_{\beta\eta} 1\to 0\to 0 <_{\beta\eta} 1\to 1\to 0\to 0 <_{\beta\eta} 3\to 0\to 0 <_{\beta\eta} 1_2\to 0\to 0$$

Terms (for these only finitely many bound variables are needed)

$$\lambda x^0.x \quad : \quad 1 \qquad \lambda x_1^0 x_2^0.x_1 : 1_2 \qquad \lambda x_1^0 \ldots x_k^0.x_i : 1_k$$

$$\lambda f^1 x^0.\underline{f(f(f}\ x)) \quad : \quad \omega \qquad \lambda f^1 g^1 x^0.\underline{f(g(g(f(f}x)))) : \omega + 1$$

$$\lambda F^{frm-e}.F(\lambda x_1^0.F(\lambda x_2^0.F(\lambda x_3^0.x_1))) \quad : \quad 3$$

$$\lambda p^{1\,frm-e} c^0.pc(pcc) \quad : \quad \top$$

DEFINITION. A type A (its inhabitants) is called *finitely generated* if $\Lambda^{\emptyset}(A)$ can be obtained from finitely many closed terms $\{M_1, \ldots, M_n\}$, not necessarily of type A, by application alone.

EXAMPLE. Let $\omega = (0\to0)\to0\to0$. Then this type is finitely generated by $\{c_0, \mathsf{S}^+\}$, where $\mathsf{S}^+ \sim_{\beta\eta} \lambda nfx.f(nfx)$.

$\lambda^0_\to$ The 'monster' type M (Statman)

$$\mathsf{M} \;:=\; 3 \to 1 \;=\; (((0 \to 0) \to 0) \to 0) \to 0 \to 0$$

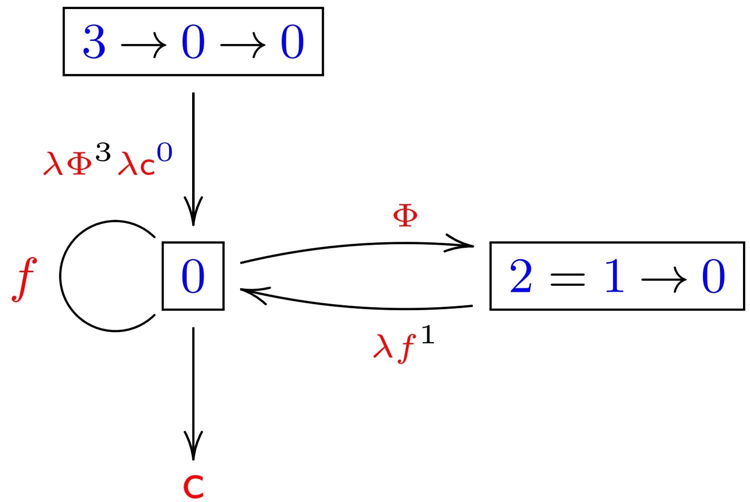
Examples of terms of this type

$\lambda\Phi^3 \mathsf{c}^0.\mathsf{c}$
$\lambda\Phi^3 \mathsf{c}^0.\Phi(\lambda f^1.f\mathsf{c})$
$\lambda\Phi^3 \mathsf{c}^0.\Phi(\lambda f_1^1.\Phi(\lambda f_2^1.f_2(f_1\mathsf{c})))$

$\lambda\Phi^3 \mathsf{c}^0.\Phi(\lambda f_1^1.\underline{f_1}(\Phi(\lambda f_2^1.\underline{f_2(f_1}(\Phi(\lambda f_3^1.\underline{f_1(f_3(f_2}\mathsf{c})))))))$

$\lambda\Phi^3 \mathsf{c}^0.\Phi(\lambda f_1^1.\vec{w}_1(\Phi(\lambda f_2^1.\vec{w}_2(\ldots\Phi(\lambda f_n^1.\vec{w}_n\mathsf{c})))) = \text{``}\langle\vec{w}_1;\ldots;\vec{w}_n\rangle\text{''},$

with $\vec{w}_k \in \{f_1,\ldots,f_k\}^*$.



'Inhabitation machine' for M

Terms of this type need arbitrarily many (bound) variables.

We say that the type M is *rich*. The types $1, \omega, \omega{+}1, \top$ are *poor*.