Chapter 16

Basic Properties 31.10.2006:581

This Chapter is on type theories but, by Remark 15.4.7, applies as well to type structures. That is, everywhere $\mathcal{T}$, TT and TT $^\top$ may be replaced by $\mathcal{S}$, TS and TS$^\top$, respectively.

Let $\mathcal{T}$ be a type theory. We derive properties of $\vdash^{\mathcal{T}}$, where $\vdash^{\mathcal{T}}$ stands for $\vdash^{\mathcal{T}}_{\cap}$ or $\vdash^{\mathcal{T}}_{\cap^\top}$. Whenever we need to require extra properties about $\mathcal{T}$, this will be stated explicitly. Often $\mathcal{T}$ will be one of the theories from Figure 15.2.

The properties that will be studied are inversion theorems that will make it possible to predict when statements

$$\Gamma \vdash^{\mathcal{T}} M : A \tag{1}$$

are derivable, in particular from what other statments. This will be done in Section 16.1. Building upon this, in Section 16.2 conditions are given when type assignment statements remain valid after reducing or expanding the M according to $\boldsymbol{\beta}$ or $\boldsymbol{\eta}$-rules.

16.1. Inversion theorems

In the style of Coppo et al. [1984] and Alessi et al. [2003], [2005] we shall isolate special properties which allow to 'reverse' some of the rules of the type assignment system $\vdash^{\mathcal{T}}_{\cap}$, thereby achieving some form of 'generation' and 'inversion' properties. These state necessary and sufficient conditions when an assertion $\Gamma \vdash^{\mathcal{T}} M : A$ holds depending on the form of M and A, see Theorems 16.1.1 and 16.1.10.

16.1.1. Theorem (Inversion Theorem I). *If $\vdash$ is $\vdash^{\mathcal{T}}_{\cap}$, then the following statements hold unconditionally; if it is $\vdash^{\mathcal{T}}_{\cap^\top}$, then they hold under the assumption that*

$A \neq \top$ in (i) and (ii).

$$\begin{array}{lrcl}
\text{(i)} & \Gamma \vdash x : A & \Leftrightarrow & \Gamma(x) \leq A. \\
\text{(ii)} & \Gamma \vdash MN : A & \Leftrightarrow & \exists k \geq 1\, \exists B_1, \ldots, B_k, C_1, \ldots, C_k \\
& & & [C_1 \cap \ldots \cap C_k \leq A \;\&\; \forall i \in \{1, \ldots, k\} \\
& & & \Gamma \vdash M : B_i \to C_i \;\&\; \Gamma \vdash N : B_i]. \\
\text{(iii)} & \Gamma \vdash \lambda x.M : A & \Leftrightarrow & \exists k \geq 1\, \exists B_1, \ldots, B_k, C_1, \ldots, C_k \\
& & & [(B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \leq A \\
& & & \;\&\; \forall i \in \{1, \ldots, k\}.\Gamma, x{:}B_i \vdash M : C_i].
\end{array}$$

Proof. We only treat ($\Rightarrow$) in (i)-(iii), as ($\Leftarrow$) is trivial. Let first $\vdash$ be $\vdash^{\mathcal{T}}_{\cap}$.

(i) By induction on derivations. We reason according which axiom or rule has been used in the last step. Only axiom (Ax), and rules ($\cap$I), ($\leq$) could have been applied. In the first case one has $\Gamma(x) \equiv A$. In the other two cases the induction hypothesis applies.

(ii) By induction on derivations. By assumption on A and the shape of the term the last applied step has to be rule ($\to$E), ($\leq$) or ($\cap$I). In the first case the last applied rule is

$$(\to\text{E}) \quad \frac{\Gamma \vdash M : D \to A \quad \Gamma \vdash N : D}{\Gamma \vdash MN : A}.$$

We can take $k = 1$ and $C_1 \equiv A$ and $B_1 \equiv D$. In the second case the last rule applied is

$$(\leq) \quad \frac{\Gamma \vdash MN : B \quad B \leq A}{\Gamma \vdash MN : A}$$

and the induction hypothesis applies. In the last case $A \equiv A_1 \cap A_2$ and the last applied rule is

$$(\cap\text{I}) \quad \frac{\Gamma \vdash MN : A_1 \quad \Gamma \vdash MN : A_2}{\Gamma \vdash MN : A_1 \cap A_2}.$$

By the induction hypothesis there are B_i, C_i, D_j, E_j, with $1 \leq i \leq k,\ 1 \leq j \leq k'$, such that

$$\begin{array}{ll}
\Gamma \vdash M : B_i \to C_i, & \Gamma \vdash N : B_i, \\
\Gamma \vdash M : D_j \to E_j, & \Gamma \vdash N : D_j, \\
C_1 \cap \ldots \cap C_k \leq A_1, & E_1 \cap \ldots \cap E_{k'} \leq A_2.
\end{array}$$

Hence we are done, as $C_1 \cap \ldots \cap C_k \cap E_1 \cap \ldots \cap E_{k'} \leq A$.

(iii) Again by induction on derivations. We only treat the case $A \equiv A_1 \cap A_2$ and the last applied rule is ($\cap$I):

$$(\cap\text{I}) \quad \frac{\Gamma \vdash \lambda x.M : A_1 \quad \Gamma \vdash \lambda x.M : A_2}{\Gamma \vdash \lambda x.M : A_1 \cap A_2}.$$

By the induction hypothesis there are B_i, C_i, D_j, E_j with $1 \leq i \leq k,\ 1 \leq j \leq k'$ such that

$$\begin{array}{ll}
\Gamma, x{:}B_i \vdash M : C_i, & (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \leq A_1, \\
\Gamma, x{:}D_j \vdash M : E_j, & (D_1 \to E_1) \cap \ldots \cap (D_{k'} \to E_{k'}) \leq A_2.
\end{array}$$

We are done, since $(B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \cap (D_1 \to E_1) \cap \ldots \cap (D_{k'} \to E_{k'}) \leq A$.

Now we prove ($\Rightarrow$) in (i)-(iii) for $\lambda^{\mathcal{T}}_{\cap\top}$.

(i) The condition $A \neq \top$ implies that axiom ($\top$ universal) cannot have been used in the last step. Hence the reasoning above suffices.

(ii), (iii) The only interesting rule is ($\cap$I). Condition $A \neq \top$ implies that we cannot have $A_1 = A_2 = \top$. In case $A_1 \neq \top$ and $A_2 \neq \top$ the result follows as above. The other cases are more easy. ∎

Notice that as a consequence of this theorem the *subformula property* holds for all $\lambda^{\mathcal{T}}_{\cap(\top)}$.

16.1.2. Corollary (Subformula property). *Assume* $\Gamma \vdash^{\mathcal{T}}_{\cap(\top)} M : A$ *and let* N *be a subterm of* M*. Then* N *is typable in an extension* $\Gamma^+ = \Gamma, x_1{:}B_1, \dots, x_n{:}B_n$ *in which also the variables* $\{x_1, \dots, x_n\} = \mathrm{FV}(N) - \mathrm{FV}(M)$ *get a type assigned.*

Proof. If we have rule ($\top$-universal) the statement is trivial. Otherwise if N is a subterm of M, then we can write $M \equiv C[N]$. The statement is proved by induction on the structure of $C[\ ]$. ∎

16.1.3. Proposition. *We have for fresh* y $(\notin \mathrm{dom}(\Gamma))$ *the following.*

$$\exists B\,[\Gamma \vdash N : B \;\&\; \Gamma \vdash M[x{:}= N] : A] \;\Rightarrow\; \exists B\,[\Gamma \vdash N : B \;\&\; \Gamma, y{:}B \vdash M[x{:}= y] : A].$$

Proof. By induction on the structure of M. ∎

Under some conditions (that will hold for many TTs, notably the ones introduced in Section 15.1), the Inversion Theorem can be restated in a more memorable form. This will be done in Theorem 16.1.10.

16.1.4. Definition. $\mathcal{T}$ is called ***β**-sound* if

$\forall k{\geq}1 \forall A_1, \dots, A_k, B_1, \dots, B_k, C, D.$

$$\left.\begin{array}{l}(A_1{\to}B_1) \cap \dots \cap (A_k{\to}B_k) \leq (C{\to}D) \;\&\; D \neq \top \;\Rightarrow\\ C \leq A_{i_1} \cap \dots \cap A_{i_p} \;\&\; B_{i_1} \cap \dots \cap B_{i_p} \leq D,\\ \text{for some } p \geq 1 \text{ and } 1 \leq i_1, \dots, i_p \leq k.\end{array}\right\} \qquad (*)$$

This definition immediately translates to type structures. The notion of **β**-soundness is introduced to prove invertibility of the rule ($\to$I), which is important for the next section.

16.1.5. Lemma. *Let* $\mathcal{T}$ *satisfy* ($\top$) *and* ($\top\!\to$)*. Suppose moreover that* $\mathcal{T}$ *is* ***β**-sound. Then for all* A, B

$$A{\to}B = \top \;\Leftrightarrow\; B = \top.$$

Proof. ($\Rightarrow$) $\top{\to}\top \leq \top = A{\to}B$, by assumption; hence $\top \leq B$ ($\leq \top$), by **β**-soundness. ($\Leftarrow$) By rule ($\top\!\to$). ∎

Let $\mathcal{T}$ be **β**-sound. Then $A{\to}B \leq A'{\to}B' \;\Rightarrow\; A' \leq A \;\&\; B \leq B'$ if B' is not the top element (but not in general).

In 16.1.6-16.1.8 we will show that all $\mathcal{T}$'s of Figures 15.2 are **β**-sound.

16.1.6. REMARK. Note that in a TT every type A can be written uniquely, apart from the order, as

$$A \equiv \alpha_1 \cap \ldots \cap \alpha_n \cap (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \qquad (+),$$

i.e. an intersection of atoms ($\alpha_i \in \mathbb{A}$) and arrow types.

For some of our $\mathcal{T}$ the shape (+) in Remark 16.1.6 can be simplified.

16.1.7. DEFINITION. For the type theories $\mathcal{T}$ of Figure 15.2 we define for each $A \in \mathbb{T}^{\mathcal{T}}$ its *canonical form*, notation $\mathrm{cf}(A)$, as follows.

(i) If $\mathcal{T} \in \{\mathrm{BCD}, \mathrm{AO}, \mathrm{Plotkin}, \mathrm{Engeler}, \mathrm{CDS}, \mathrm{CDV}, \mathrm{CD}\}$, then

$$\mathrm{cf}(A) \equiv A.$$

(ii) If $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{HL}\}$ then the definition is by induction on A. For an atom α the canonical form $\mathrm{cf}(\alpha)$ depends on the type theory in question; moreover the mapping cf preserves $\to, \cap$ and $\top$.

System $\mathcal{T}$	A	$\mathrm{cf}(A)$
Scott	ω	$\top \to \omega$
Park	ω	$\omega \to \omega$
CDZ, HL	ω	$\varphi \to \omega$
	φ	$\omega \to \varphi$
HR	ω	$\varphi \to \omega$
	φ	$(\omega \to \omega) \cap (\varphi \to \varphi)$
DHM	φ	$\omega \to \varphi$
	ω	$\top \to \varphi$
All systems except HL	$\top$	$\top$
All systems	$B \to C$	$B \to C$
All systems	$B \cap C$	$\mathrm{cf}(B) \cap \mathrm{cf}(C)$

16.1.8. THEOREM. *All theories $\mathcal{T}$ of Figure 15.2 are $\boldsymbol{\beta}$-sound.*

PROOF. We prove the following stronger statement (induction loading). Let

$$\begin{array}{rcl} A & \leq & A', \\ \mathrm{cf}(A) & \equiv & \alpha_1 \cap \ldots \cap \alpha_n \cap (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k), \\ \mathrm{cf}(A') & \equiv & \alpha'_1 \cap \ldots \cap \alpha'_{n'} \cap (B'_1 \to C'_1) \cap \ldots \cap (B'_{k'} \to C'_{k'}). \end{array}$$

Then

$$\forall j \in \{1, k'\}.[C_{j'} \neq \top \Rightarrow$$
$$\exists p \geq 1 \exists i_1, \ldots i_p \in \{1, k\}.[B'_j \leq B_{i_1} \cap \ldots \cap B_{i_p} \ \&\ C_{i_1} \cap \ldots \cap C_{i_p} \leq C'_j]].$$

The proof of the statement is by induction on the generation of $A \leq A'$. From it $\boldsymbol{\beta}$-soundness follows easily. ■

16.1.9. Remark. From the Theorem it follows immediately that for the compatible theories of Fig. 15.2 the corresponding type structures are $\boldsymbol{\beta}$-sound.

16.1.10. Theorem (Inversion Theorem II). *Of the following properties (i) holds in general, (ii) provided that $\mathcal{T}$ is proper and $A \neq \top$ if $\vdash$ is $\vdash^{\mathcal{T}}_{\cap\top}$ and (iii) provided that $\mathcal{T}$ is $\boldsymbol{\beta}$-sound.*

$$\begin{array}{rrcl}
\text{(i)} & \Gamma, x{:}A \vdash x : B & \Leftrightarrow & A \leq B. \\
\text{(ii)} & \Gamma \vdash (MN) : A & \Leftrightarrow & \exists B\, [\Gamma \vdash M : (B{\to}A)\ \&\ \Gamma \vdash N : B]. \\
\text{(iii)} & \Gamma \vdash (\lambda x.M) : (B{\to}C) & \Leftrightarrow & \Gamma, x{:}B \vdash M : C.
\end{array}$$

Proof. The proof of each ($\Leftarrow$) is easy. So we only treat ($\Rightarrow$).

(i) If $B \neq \top$, then the conclusion follows from Theorem 16.1.1(i). If $B = \top$, then the conclusion holds trivially.

(ii) Suppose $\Gamma \vdash MN : A$. Then by Theorem 16.1.1(ii) there are $B_1, \ldots, B_k$, $C_1, \ldots, C_k$, with $k \geq 1$, such that $C_1 \cap \ldots \cap C_k \leq A$, $\Gamma \vdash M : B_i{\to}C_i$ and $\Gamma \vdash N : B_i$ for $1 \leq i \leq k$. Hence $\Gamma \vdash N : B_1 \cap \ldots \cap B_k$ and

$$\begin{array}{rl}
\Gamma \vdash M : & (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \\
& \leq (B_1 \cap \ldots \cap B_k){\to}(C_1 \cap \ldots \cap C_k) \\
& \leq (B_1 \cap \ldots \cap B_k){\to}A,
\end{array}$$

by Lemma 15.1.13. So we can take $B \equiv (B_1 \cap \ldots \cap B_k)$.

(iii) Suppose $\Gamma \vdash (\lambda x.M) : (B{\to}C)$. Then Theorem 16.1.1(iii) applies and we have for some $k \geq 1$ and $B_1, \ldots, B_k, C_1, \ldots, C_k$

$$\begin{array}{c}
(B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \leq B{\to}C, \\
\Gamma, x{:}B_i \vdash M : C_i \text{ for all } i.
\end{array}$$

If $C = \top$, then the assertion holds trivially, so let $C \neq \top$. Then by $\boldsymbol{\beta}$-soundness there are $1 \leq i_1, \ldots, i_p \leq k$, $p \geq 1$ such that

$$\begin{array}{c}
B \leq B_{i_1} \cap \ldots \cap B_{i_p}, \\
C_{i_1} \cap \ldots \cap C_{i_p} \leq C.
\end{array}$$

Applying ($\leq$-L) we get

$$\begin{array}{c}
\Gamma, x{:}B \vdash M : C_{i_j},\ 1 \leq j \leq p, \\
\Gamma, x{:}B \vdash M : C_{i_1} \cap \ldots \cap C_{i_p} \leq C. \blacksquare
\end{array}$$

We give a simple example which shows that in general rule ($\to$E) cannot be reversed, i.e. that if $\Gamma \vdash MN : B$, then it is not always true that there exists A such that $\Gamma \vdash M : A{\to}B$ and $\Gamma \vdash N : A$.

16.1.11. Example. Let $\mathcal{T}$ = Engeler, one of the intersection type theories of Figure 15.2. Let $\Gamma = \{x{:}(\varphi_0{\to}\varphi_1) \cap (\varphi_2{\to}\varphi_3), y{:}(\varphi_0 \cap \varphi_2)\}$. Then one has $\Gamma \vdash^{\mathcal{T}}_{\cap\top} xy : \varphi_1 \cap \varphi_3$. Nevertheless, it is not possible to find a type B such that $\Gamma \vdash^{\mathcal{T}}_{\cap\top} x : B{\to}(\varphi_1 \cap \varphi_3)$ and $\Gamma \vdash^{\mathcal{T}}_{\cap\top} y : B$. See Exercise ??.

16.1.12. Remark. In general

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : A \not\Rightarrow \exists B, C. A = (B{\to}C) \;\&\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : C.$$

A counterexample is $\vdash^{\mathrm{BCD}} \mathsf{I} : (\alpha_1{\to}\alpha_1) \cap (\alpha_2{\to}\alpha_2)$, with α_1, α_2 atomic.

16.1.13. Proposition. *For $\mathcal{T} \in \{$Scott, Park, CDZ, HR, DHM, BCD, AO$\}$ the properties (i), (ii) and (iii) of Theorem 16.1.10 hold for $\vdash^{\mathcal{T}}_{\cap^\top}$, provided that in (ii) $A \neq \top$ for $\mathcal{T} =$ AO. For $\mathcal{T} \in \{$HL, CDV$\}$ the properties hold unconditionally for $\vdash^{\mathcal{T}}_{\cap}$.*

Proof. For these $\mathcal{T}$ Theorem 16.1.10 applies since they are proper and $\boldsymbol{\beta}$-sound (by Theorem 16.1.8). Moreover, by axiom $(\to\top)$ we have $\Gamma \vdash^{\mathcal{T}}_{\cap^\top} M : \top \to \top$ for all Γ, M, hence we do not need to assume $A \neq \top$ for $\mathcal{T} \in \{$Scott, Park, CDZ, HR, DHM, BCD$\}$. ■

16.2. Subject reduction and expansion

Various subject reduction and expansion properties are proved, for the classical $\boldsymbol{\beta}$, $\boldsymbol{\beta}\mathsf{I}$ and $\boldsymbol{\eta}$ notions of reduction. Other results can be found in Alessi et al. [2003], Alessi et al. [2006]. We consider the following rules.

$$(R\text{-}red) \qquad \frac{M \to_R N \quad \Gamma \vdash M : A}{\Gamma \vdash N : A}$$

$$(R\text{-}exp) \qquad \frac{M\,{}_R{\leftarrow}\, N \quad \Gamma \vdash M : A}{\Gamma \vdash N : A}$$

where R is a notion of reduction, notably $\boldsymbol{\beta}$-, $\boldsymbol{\beta}\mathsf{I}$, or $\boldsymbol{\eta}$-reduction. If one of these rules holds in $\lambda^{\mathcal{T}}_{\cap(\top)}$, we write $\lambda^{\mathcal{T}}_{\cap(\top)} \models (R\text{-}\{exp, red\})$, respectively. If both hold we write $\lambda^{\mathcal{T}}_{\cap(\top)} \models (R\text{-}cnv)$. These properties will be crucial in Chapters 17, 18 and 19, where we will discuss (untyped) λ-models induced by these systems.

Recall that $(\lambda x.M)N$ is a $\boldsymbol{\beta}\mathsf{I}$-*redex* if $x \in \mathrm{FV}(M)$, Curry and Feys [1958].

$\boldsymbol{\beta}$-conversion

We first investigate when $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}(\mathsf{I})\text{-}red)$.

16.2.1. Proposition. (i) $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\mathsf{I}\text{-}red) \Leftrightarrow$

$$[\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}A) \;\&\; x \in \mathrm{FV}(M) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : A].$$

(ii) $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\text{-}red) \Leftrightarrow [\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}A) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : A]$.

Proof. (i) ($\Rightarrow$) Assume $\Gamma \vdash \lambda x.M : B{\to}A \;\&\; x \in \mathrm{FV}(M)$, which implies $\Gamma, y{:}B \vdash (\lambda x.M)y : A$, by weakening and rule $(\to\mathrm{E})$ for a fresh y. Now rule $(\boldsymbol{\beta}\mathsf{I}\text{-}red)$ gives us $\Gamma, y{:}B \vdash M[x{:=}y] : A$. Hence $\Gamma, x{:}B \vdash M : A$.

($\Leftarrow$) Suppose $\Gamma \vdash (\lambda x.M)N : A \;\&\; x \in \mathrm{FV}(M)$, in order to show that $\Gamma \vdash M[x{:=}N] : A$. We may assume $A \neq \top$. Then Theorem 16.1.1(ii) implies $\Gamma \vdash \lambda x.M : B_i{\to}C_i$, $\Gamma \vdash N : B_i$ and $C_1 \cap \ldots \cap C_k \leq A$, for some

$B_1, \ldots, B_k, C_1, \ldots, C_k$. By assumption $\Gamma, x{:}B_i \vdash M : C_i$. Hence by rule (*cut*), Proposition 15.2.8, one has $\Gamma \vdash M[x{:=}N] : C_i$. Therefore $\Gamma \vdash M[x{:=}N] : A$, using rules ($\cap$I) and ($\leq$).

(ii) Similarly. ■

16.2.2. Corollary. *Let $\mathcal{T}$ be $\boldsymbol{\beta}$-sound. Then* $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\textit{-red})$.

Proof. Using Theorem 16.1.10(iii). ■

16.2.3. Corollary. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{BCD}, \mathrm{AO}, \mathrm{Plotkin}, \mathrm{Engeler}, \mathrm{CDS}\}$. *Then* $\lambda^{\mathcal{T}}_{\cap\top} \models (\boldsymbol{\beta}\textit{-red})$.

(ii) *Let* $\mathcal{T} \in \{\mathrm{HL}, \mathrm{CD}, \mathrm{CDV}\}$. *Then* $\lambda^{\mathcal{T}}_{\cap} \models (\boldsymbol{\beta}\textit{-red})$.

Proof. By Corollary 16.2.2 and Theorem 16.1.8. ■

In Definition 18.2.22 we will introduce a type theory that is not $\boldsymbol{\beta}$-sound, but nevertheless induces a type assignment system satisfying ($\boldsymbol{\beta}$-*red*).
Now we investigate when $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\textit{-exp})$. As a warm-up, suppose that $\Gamma \vdash M[x{:=}N] : A$. Then we would like to conclude that N has a type, as it seems to be a subformula, and therefore $\Gamma \vdash (\lambda x.M)N : A$. There are two problems: N may occur several times in $M[x{:=}N]$, so that it has (should have) in fact several types. In the system $\lambda_{\to}$ this problem causes the failure of rule ($\boldsymbol{\beta}$-*exp*). But in the intersection type theories one has $N : B_1 \cap \ldots \cap B_k$ if $N : B_1, \ldots, N : B_k$. Therefore $(\lambda x.M)N$ has a type if $M[x{:=}N]$ has one. The second problem arises if N does not occur at all in $M[x{:=}N]$, i.e. if the redex is a λK-redex. We would like to assign as type to N the intersection over an empty sequence, i.e. the top $\top$. This makes ($\boldsymbol{\beta}$-*exp*) invalid in $\lambda^{\mathcal{T}}_{\cap}$, but valid in systems $\lambda^{\mathcal{T}}_{\cap\top}$.

16.2.4. Proposition. (i) *Suppose* $\Gamma \vdash^{\mathcal{T}} M[x{:=}N] : A$. *Then*

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M)N : A \;\Leftrightarrow\; N \textit{ is typable in context } \Gamma.$$

(ii) $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\textit{-exp}) \;\Leftrightarrow\; \forall \Gamma, M, N, A$

$$[\Gamma \vdash^{\mathcal{T}} M[x{:=}N] : A \;\Rightarrow\; N \textit{ is typable in context } \Gamma].$$

(iii) $\lambda^{\mathcal{T}}_{\cap(\top)} \models (\boldsymbol{\beta}\mathsf{I}\textit{-exp}) \;\Leftrightarrow\; \forall \Gamma, M, N, A$ *with* $x \in \mathrm{FV}(M)$

$$[\Gamma \vdash^{\mathcal{T}} M[x{:=}N] : A \;\Rightarrow\; N \textit{ is typable in context } \Gamma].$$

Proof. (i) ($\Rightarrow$) By Theorem 16.1.1(ii). ($\Leftarrow$) Let $\Gamma \vdash M[x{:=}N] : A$ and suppose N is typable in context Γ. By proposition 16.1.3 for some B and a fresh y one has $\Gamma \vdash N : B \;\&\; \Gamma, y{:}B \vdash M[x{:}=y] : A$. Then $\Gamma \vdash \lambda x.M : (B{\to}A)$ and hence $\Gamma \vdash (\lambda x.M)N : A$.

(ii) ($\Rightarrow$) Assume $\Gamma \vdash M[x{:=}N] : A$. Then $\Gamma \vdash (\lambda x.M)N : A$, by ($\boldsymbol{\beta}$-*exp*), hence by (i) we are done. ($\Leftarrow$) Assume $\Gamma \vdash L' : A$, with $L \to_{\boldsymbol{\beta}} L'$. By induction on the generation of $L \to_{\boldsymbol{\beta}} L'$ we get $\Gamma \vdash L : A$ from (i) and Theorem 16.1.1.

(iii) Similar to (ii). ■

16.2.5. Corollary. (i) $\lambda_{\cap\top}^{\mathcal{T}} \models (\boldsymbol{\beta}\text{-}exp)$.
(ii) $\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\beta}\mathsf{I}\text{-}exp)$.

Proof. (i) Trivial, since every term has type $\top$.
(ii) By the subformula property (Corollary 16.1.2). ∎

Now we can harvest results towards closure under $\boldsymbol{\beta}$-conversion.

16.2.6. Theorem. *Let $\mathcal{T} \in \mathrm{TT}$ be $\boldsymbol{\beta}$-sound.*
(i) *Let $\mathcal{T} \in \mathrm{TT}^{\top}$. Then $\lambda_{\cap\top}^{\mathcal{T}} \models (\boldsymbol{\beta}\text{-}cnv)$.*
(ii) $\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\beta}\mathsf{I}\text{-}cnv)$.

Proof. (i) By Corollaries 16.2.2 and 16.2.5(i).
(ii) By Corollaries 16.2.2 and 16.2.5(ii). ∎

16.2.7. Corollary. (i) *Let $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{BCD}, \mathrm{AO}, \mathrm{Plotkin}, \mathrm{Engeler}, \mathrm{CDS}\}$. Then $\lambda_{\cap\top}^{\mathcal{T}} \models (\boldsymbol{\beta}\text{-}cnv)$.*
(ii) *Let $\mathcal{T} \in \{\mathrm{HL}, \mathrm{CDV}, \mathrm{CD}\}$. Then $\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\beta}\mathsf{I}\text{-}cnv)$.*

Proof. (i) By Theorem 16.2.6(i).
(ii) By Theorem 16.2.6(ii). ∎

$\boldsymbol{\eta}$-conversion

First we give necessary and sufficient conditions for a system $\lambda_{\cap(\top)}^{\mathcal{T}}$ to satisfy the rule ($\boldsymbol{\eta}$-*red*).

16.2.8. Theorem. (i) *Let $\mathcal{T} \in \mathrm{TT}^{\top}$. Then*

$$\lambda_{\cap\top}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}red) \;\Leftrightarrow\; \mathcal{T} \text{ is natural.}$$

(ii) *Let $\mathcal{T} \in \mathrm{TT}$. Then*

$$\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}red) \;\Leftrightarrow\; \mathcal{T} \text{ is proper.}$$

Proof. (i) ($\Rightarrow$) Assume $\lambda_{\cap\top}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}red)$ towards $(\to\cap)$, $(\to)$ and $(\top\to)$.

As to $(\to\cap)$, one has

$$x{:}(A\to B) \cap (A\to C), y{:}A \vdash xy : B \cap C,$$

hence by $(\to\mathrm{I})$ it follows that $x{:}(A\to B) \cap (A\to C) \vdash \lambda y.xy : A\to(B \cap C)$. Therefore $x{:}(A\to B) \cap (A\to C) \vdash x : A\to(B \cap C)$, by ($\boldsymbol{\eta}$-*red*). By Theorem 16.1.10(i) one can conclude $(A\to B) \cap (A\to C) \leq A\to(B \cap C)$.

As to $(\to)$, suppose that $A \leq B$ and $C \leq D$, in order to show $B\to C \leq A\to D$. One has $x{:}B\to C, y{:}A \vdash xy : C \leq D$, so $x{:}B\to C \vdash \lambda y.xy : A\to D$. Therefore by ($\boldsymbol{\eta}$-*red*) it follows that $x{:}B\to C \vdash x : A\to D$ and we are done as before.

As to $\top \leq \top\to\top$, notice that $x{:}\top, y{:}\top \vdash xy : \top$, so we have $x{:}\top \vdash \lambda y.xy : \top\to\top$. Therefore $x{:}\top \vdash x : \top\to\top$ and again we are done.

($\Leftarrow$) Let $\mathcal{T}$ be natural. Assume that $\Gamma \vdash \lambda x.Mx : A$, with $x \notin \mathrm{FV}(M)$, in order to show $\Gamma \vdash M : A$. If $A = \top$, we are done. Otherwise,

$$\begin{array}{lll} \Gamma \vdash \lambda x.Mx : A & \Rightarrow & \Gamma, x{:}B_i \vdash Mx : C_i, 1 \le i \le k, \ \& \\ & & (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \le A, \\ & & \text{for some } B_1, \ldots, B_k, C_1, \ldots, C_k, \end{array}$$

by Theorem 16.1.1(iii). By Lemma 16.1.5 we omit the i such that $C_i = \top$. There is at least one $C_i \neq \top$, since otherwise $A \ge (B_1 \to \top) \cap \ldots \cap (B_k \to \top) = \top$, again by Lemma 16.1.5, and we would have $A = \top$. Hence by Theorem 16.1.10(ii)

$$\begin{array}{lll} \Rightarrow & \Gamma, x{:}B_i \vdash M : D_i \to C_i \text{ and} & \\ & \Gamma, x{:}B_i \vdash x : D_i, & \text{for some } D_1, \ldots, D_k, \\ \Rightarrow & B_i \le D_i, & \text{by Theorem 16.1.10(i)}, \\ \Rightarrow & \Gamma \vdash M : (B_i \to C_i), & \text{by } (\le\text{-L}) \text{ and } (\to), \\ \Rightarrow & \Gamma \vdash M : ((B_1 \to C_1) \cap \ldots \cap (B_k \to C_k)) \le A. & \end{array}$$

(ii) Similarly, but simpler. ∎

16.2.9. Corollary. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{BCD}\}$. *Then* $\lambda_{\cap^\top}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}red)$.

(ii) *Let* $\mathcal{T} \in \{\mathrm{HL}, \mathrm{CDV}\}$. *Then* $\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}red)$.

In order to characterize the admissibility of rule ($\boldsymbol{\eta}$-*exp*), we need to introduce a further condition on type theories. This condition is necessary and sufficient to derive from the basis $x{:}A$ the same type A for $\lambda y.xy$, as we will show in the proof of Theorem 16.2.11.

16.2.10. Definition. Let $\mathcal{T} \in \mathrm{TT}$.
(i) $\mathcal{T}$ is called $\boldsymbol{\eta}$-*sound* iff for all A there are $k \ge 1$, $m_1, \ldots, m_k \ge 1$ and $B_1, \ldots, B_k$, $C_1, \ldots, C_k$,

$$\begin{pmatrix} D_{11} \ldots D_{1m_1} \\ \ldots \\ D_{k1} \ldots D_{km_k} \end{pmatrix} \text{ and } \begin{pmatrix} E_{11} \ldots E_{1m_1} \\ \ldots \\ E_{k1} \ldots E_{km_k} \end{pmatrix}$$

with

$$\begin{array}{l} (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \ \le \ A \\ \& \ A \ \le \ (D_{11} \to E_{11}) \cap \ldots \cap (D_{1m_1} \to E_{1m_1}) \cap \\ \qquad \ldots \\ \qquad (D_{k1} \to E_{k1}) \cap \ldots \cap (D_{km_k} \to E_{km_k}) \\ \& \ B_i \le D_{i1} \cap \ldots \cap D_{im_i} \ \& \ E_{i1} \cap \ldots \cap E_{im_i} \le C_i, \\ \quad \text{for } 1 \le i \le k. \end{array}$$

(ii) Let $\mathcal{T} \in \mathrm{TT}^\top$. Then $\mathcal{T}$ is called $\boldsymbol{\eta}^\top$-*sound* iff for all $A \neq \top$ at least one of the following two conditions holds.

(1) There are types $B_1, \ldots, B_n$ with $(B_1 \to \top) \cap \ldots \cap (B_n \to \top) \le A$;

(2) There are $k \geq 1$, $m_1, \ldots, m_k \geq 1$ and $B_1, \ldots, B_k$, $C_1, \ldots, C_k$,

$$\begin{pmatrix} D_{11} \ldots D_{1m_1} \\ \ldots \\ D_{k1} \ldots D_{km_k} \end{pmatrix} \text{ and } \begin{pmatrix} E_{11} \ldots E_{1m_1} \\ \ldots \\ E_{k1} \ldots E_{km_k} \end{pmatrix}$$

with

$$\begin{array}{l} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \cap \\ \cap\, (B_{k+1}{\to}\top) \cap \ldots \cap (B_n{\to}\top) \;\; \leq \;\; A \\ \&\;\; A \;\; \leq \;\; (D_{11}{\to}E_{11}) \cap \ldots \cap (D_{1m_1}{\to}E_{1m_1}) \cap \\ \qquad\qquad \ldots \\ \qquad\qquad (D_{k1}{\to}E_{k1}) \cap \ldots \cap (D_{km_k}{\to}E_{km_k}) \\ \&\;\; B_i \leq D_{i1} \cap \ldots \cap D_{im_i} \;\&\; E_{i1} \cap \ldots \cap E_{im_i} \leq C_i, \\ \quad\; \text{for } 1 \leq i \leq k. \end{array}$$

This definition immediately translates to type structures. The validity of $\boldsymbol{\eta}$-expansion can be given as follows.

16.2.11. Theorem (Characterization of $\boldsymbol{\eta}$-*exp*).

$$\begin{array}{llll} \text{(i)} & \lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}exp) & \Leftrightarrow & \mathcal{T} \textit{ is } \boldsymbol{\eta}\textit{-sound.} \\ \text{(ii)} & \lambda_{\cap^\top}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-}exp) & \Leftrightarrow & \mathcal{T} \textit{ is } \boldsymbol{\eta}^\top\textit{-sound.} \end{array}$$

Proof. (i) ($\Rightarrow$) Assume $\lambda_{\cap}^{\mathcal{T}} \models (\boldsymbol{\eta}\text{-exp})$. As $x{:}A \vdash x : A$, by assumption we have $x{:}A \vdash \lambda y.xy : A$. From Theorem 16.1.1(iii) it follows that $x{:}A, y{:}B_i \vdash xy : C_i$ and $(B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \leq A$ for some B_i, C_i. By Theorem 16.1.1(ii) for each i there exist D_{ij}, E_{ij}, such that for each j one has $x{:}A, y{:}B_i \vdash x : (D_{ij}{\to}E_{ij})$, $x{:}A, y{:}B_i \vdash y : D_{ij}$ and $E_{i1} \cap \ldots \cap E_{im_i} \leq C_i$. Hence by Theorem 16.1.1(i) we have $A \leq (D_{ij}{\to}E_{ij})$ and $B_i \leq D_{ij}$ for all i and j. Therefore we obtain the condition of 16.2.10(i).

($\Leftarrow$) Suppose that $\Gamma \vdash M : A$ in order to show $\Gamma \vdash \lambda x.Mx : A$, with x fresh. By assumption A satisfies the condition of Definition 16.2.10(i).

$$\begin{array}{l} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \;\; \leq \;\; A \\ \&\;\; A \;\; \leq \;\; (D_{11}{\to}E_{11}) \cap \ldots \cap (D_{1m_1}{\to}E_{1m_1}) \cap \\ \qquad\qquad \ldots \\ \qquad\qquad (D_{k1}{\to}E_{k1}) \cap \ldots \cap (D_{km_k}{\to}E_{km_k}) \\ \&\;\; B_i \leq D_{i1} \cap \ldots \cap D_{im_i} \;\&\; E_{i1} \cap \ldots \cap E_{im_i} \leq C_i, \\ \quad\; \text{for } 1 \leq i \leq k. \end{array}$$

By rule ($\leq$) for all i, j we have $\Gamma \vdash M : D_{ij}{\to}E_{ij}$ and so $\Gamma, x{:}D_{ij} \vdash Mx : E_{ij}$ by rule ($\to$E). From ($\leq$ L), ($\cap$I) and ($\leq$) we get $\Gamma, x{:}B_i \vdash Mx : C_i$ and this implies $\Gamma \vdash \lambda x.Mx : B_i{\to}C_i$, using rule ($\to$I). So we can conclude by ($\cap$I) and ($\leq$) that $\Gamma \vdash \lambda x.Mx : A$.

(ii) The proof is nearly the same as for (i). ($\Rightarrow$) Again we get $x{:}A, y{:}B_i \vdash xy : C_i$ and $(B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \leq A$ for some B_i, C_i. If all $C_i = \top$, then A satisfies the first condition of Definition 16.2.10(ii). Otherwise, consider the i such that $C_i \neq \top$ and reason as in the proof of ($\Rightarrow$) for (i).

($\Leftarrow$) Suppose that $\Gamma \vdash M : A$ in order to show $\Gamma \vdash \lambda x.Mx : A$, with x fresh. If A satisfies the first condition of Definition 16.2.10(ii), that is $(B_1 \to \top) \cap \ldots \cap (B_n \to \top) \leq A$, then by rule $(\top)$ it follows that $\Gamma, x{:}B_i \vdash Mx : \top$, hence $\Gamma \vdash \lambda x.Mx : (B_1 \to \top) \cap \ldots \cap (B_n \to \top) \leq A$. Now let A satisfy the second condition. Then the proof is similar to that for ($\Leftarrow$) in (i). ■

For most intersection type theories of interest the condition of $\boldsymbol{\eta}(^\top)$-soundness is deduced from the following proposition.

16.2.12. Proposition. *Let $\mathcal{T} \in \mathrm{TT}$ with atoms $\mathbb{A}$ be proper.*

(i) $\mathcal{T}$ *is* $\boldsymbol{\eta}$*-sound* $\Leftrightarrow$ $\forall A \in \mathbb{A}\, \exists B_1, \ldots, B_k, C_1, \ldots, C_k\, \exists n \geq 1$
$A = (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k)$.

(ii) *Let* $\mathcal{T} \in \mathrm{TT}^\top$. *Then*

$$\begin{array}{lcl} \mathcal{T} \textit{ is } \boldsymbol{\eta}^\top\textit{-sound} & \Leftrightarrow & \forall A \in \mathbb{A}[\top \to \top \leq A \vee \exists B_1, \ldots, B_k, C_1, \ldots, C_k \\ \exists k \geq 1 & & [(B_1 \to C_1) \cap \ldots \cap (B_k \to C_k) \cap (\top \to \top) \leq A \\ & & \;\&\; A \leq (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k)]]. \end{array}$$

(iii) *Let* $\mathcal{T} \in NTT^\top$. *Then*

$$\mathcal{T} \textit{ is } \boldsymbol{\eta}^\top\textit{-sound} \;\Leftrightarrow\; \mathcal{T} \textit{ is } \boldsymbol{\eta}\textit{-sound.}$$

Proof. (i) ($\Rightarrow$) Suppose $\mathcal{T}$ is $\boldsymbol{\eta}$-sound. Let $A \in \mathbb{A}$. Then A satisfies the condition of Definition 16.2.10(i), for some $B_1, \ldots, B_k$, $C_1, \ldots, C_k$, $D_{11}, \ldots, D_{1m_1}, \ldots, D_{k1}, \ldots, D_{km_1}$, $E_{11}, \ldots, E_{1m_1}, \ldots, E_{k1}, \ldots, E_{km_k}$. By $(\to\cap)$ and $(\to)$, using Proposition 15.1.13, it follows that

$$\begin{array}{rcl} A & \leq & (D_{11} \cap \ldots \cap D_{1m_1} \to E_{11} \cap \ldots \cap E_{1m_1}) \cap \ldots \cap \\ & & (D_{k1} \cap \ldots \cap D_{km_k} \to E_{k1} \cap \ldots \cap E_{km_k}) \\ & \leq & (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k), \end{array}$$

hence $A =_{\mathcal{T}} (B_1 \to C_1) \cap \ldots \cap (B_k \to C_k)$.

($\Leftarrow$) By induction on the generation of A one can show that A satisfies the condition of $\boldsymbol{\eta}$-soundness. The case $A_1 \to A_2$ is trivial and the case $A \equiv A_1 \cap A_2$ follows by the induction hypothesis and Rule (mon).

(ii) Similarly. Note that $(\top \to \top) \leq (B \to \top)$ for all B.

(iii) Immediately by (ii) using rule $(\top\to)$. ■

16.2.13. Corollary. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{AO}\}$. *Then* $\mathcal{T}$ *is* $\boldsymbol{\eta}^\top$*-sound.*

(ii) HL *is* $\boldsymbol{\eta}$*-sound.*

Proof. Easy. For AO in (i) one applies (ii) of the Proposition. ■

16.2.14. Corollary. (i) *Let* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{AO}\}$. *Then*

$$\lambda_{\cap^\top}^{\mathcal{T}} \models (\boldsymbol{\eta}\textit{-exp}).$$

(ii) *Let* $\mathcal{T} = \mathrm{HL}$, *then*

$$\lambda_\cap^\mathcal{T} \models (\boldsymbol{\eta}\text{-}exp).$$

PROOF. By the previous Corollary and Theorem 16.2.11. ■

Exercise 16.3.15 shows that the remaining systems of Figure 15.2 do not satisfy $(\boldsymbol{\eta}\text{-}exp)$.

Now we can harvest results towards closure under $\boldsymbol{\eta}$-conversion.

$\mathcal{T}$	$\boldsymbol{\beta}$-red	$\boldsymbol{\beta}$I-red	$\boldsymbol{\beta}$-exp	$\boldsymbol{\beta}$I-exp	$\boldsymbol{\eta}$-red	$\boldsymbol{\eta}$-exp
Scott	√	√	√	√	√	√
Park	√	√	√	√	√	√
CDZ	√	√	√	√	√	√
HR	√	√	√	√	√	√
DHM	√	√	√	√	√	√
BCD	√	√	√	√	√	✗
AO	√	√	√	√	✗	√
Plotkin	√	√	√	√	✗	✗
Engeler	√	√	√	√	✗	✗
CDS	√	√	√	√	✗	✗
HL	√	√	✗	√	√	√
CDV	√	√	✗	√	√	✗
CD	√	√	✗	√	✗	✗

Figure 16.1: Type theories versus reduction and expansion

16.2.15. THEOREM. (i) *Let* $\mathcal{T} \in \mathrm{TT}^\top$. *Then*

$$\lambda_{\cap^\top}^\mathcal{T} \models (\boldsymbol{\eta}\text{-}cnv) \;\Leftrightarrow\; \mathcal{T} \textit{ is natural and } \boldsymbol{\eta}^\top\textit{-sound.}$$

(ii) *Let* $\mathcal{T} \in \mathrm{TT}$. *Then*

$$\lambda_\cap^\mathcal{T} \models (\boldsymbol{\eta}\text{-}cnv) \;\Leftrightarrow\; \mathcal{T} \textit{ is proper and } \boldsymbol{\eta}\textit{-sound.}$$

PROOF. (i) By Theorems 16.2.11(ii) and 16.2.8(i).

(ii) By Theorems 16.2.11(i) and 16.2.8(ii). ■

16.2.16. THEOREM. (i) *For* $\mathcal{T} \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}\}$ *one has*

$$\lambda_{\cap^\top}^\mathcal{T} \models (\boldsymbol{\eta}\text{-}cnv).$$

(ii) *For* $\mathcal{T} = \mathrm{HL}$ *one has*

$$\lambda_\cap^\mathcal{T} \models (\boldsymbol{\eta}\text{-}cnv).$$

PROOF. (i) By Corollaries 16.2.9(i) and 16.2.14(i).

(ii) By Corollaries 16.2.9(ii) and 16.2.14(ii). ■

Figure 16.1 summarises the results of this section and of the exercises in the following section for the type theories of Figure 15.2. The symbol '√' stands for "the property holds" and '✗' for "the property fails".

16.3. Exercises

16.3.1. Show that for each number $n \in \mathbb{N}$ there is a type $A_n \in \mathbb{T}^{\mathrm{CD}}$ such that for the Church numerals $\mathbf{c}_n$ one has $\Gamma \vdash_{\cap}^{\mathrm{CD}} \mathbf{c}_{n+1} : A_n$, but $\Gamma \nvdash_{\cap}^{\mathrm{CD}} \mathbf{c}_n : A_n$.

16.3.2. Show that $\mathsf{S}(\mathsf{KI})(\mathsf{II})$ and $(\lambda x.xxx)\mathsf{S}$ are typable in $\vdash_{\cap}^{\mathrm{CD}}$.

16.3.3. Derive $\vdash_{\cap^\top}^{\mathrm{CDZ}} (\lambda x.xxx)\mathsf{S} : \varphi$ and $y{:}\omega, z{:}\omega \vdash_{\cap^\top}^{\mathrm{CDZ}} (\lambda x.xxx)(\mathsf{S}yz) : \omega$.

16.3.4. Find the relation between the following three types w.r.t. $\leq_{\mathrm{CDZ}}$.

$$(\omega \to (\varphi \to \varphi) \to \omega) \cap ((\varphi \to \varphi) \to \varphi), (\omega \to \omega) \to \omega \text{ and } \varphi \to (\omega \to \omega) \to \varphi.$$

16.3.5. Using the Inversion Theorems show the following.

(i) $\nvdash_{\cap}^{\mathrm{CD}} \mathsf{1} : \alpha \to \alpha$, where α is any constant.

(ii) $\nvdash_{\cap}^{\mathrm{HL}} \mathsf{K} : \omega$.

(iii) $\nvdash_{\cap^\top}^{\mathrm{Scott}} \mathsf{I} : \omega$.

(iv) $\nvdash_{\cap^\top}^{\mathrm{Plotkin}} \mathsf{I}x : \omega$.

16.3.6. We say that M and M' have the same types in Γ, notation $M \sim_\Gamma M'$ if

$$\forall A\, [\Gamma \vdash M : A \;\Leftrightarrow\; \Gamma \vdash M' : A].$$

Prove that $M \sim_\Gamma M' \;\Rightarrow\; M\vec{N} \sim_\Gamma M'\vec{N}$.

16.3.7. Show that $\mathcal{T}$ = Plotkin is $\boldsymbol{\beta}$-sound by checking that it satisfies the following stronger condition.

$$(A_1 \to B_1) \cap \ldots \cap (A_n \to B_n) \leq C \to D \;\Rightarrow$$
$$\exists k \neq 0\, \exists i_1, \ldots, i_k . 1 \leq i_j \leq n \;\&\; C = A_{i_j} \;\&\; B_{i_1} \cap \ldots \cap B_{i_k} = D.$$

16.3.8. Show that $\mathcal{T}$ = Engeler is $\boldsymbol{\beta}$-sound by checking that it satisfies the following stronger condition:

$$(A_1 \to B_1) \cap \ldots \cap (A_n \to B_n) \leq C \to D \& D \neq \top \;\Rightarrow$$
$$\exists k \neq 0\, \exists i_1, \ldots, i_k . 1 \leq i_j \leq n \;\&\; C = A_{i_j} \;\&\; B_{i_1} \cap \ldots \cap B_{i_k} = D.$$

16.3.9. Let $\mathbb{A}^{\mathcal{T}} = \{\top, \omega\}$ and $\mathcal{T}$ be defined by the axioms and rules of the theories Scott and Park together. Show that $\mathcal{T}$ is not $\boldsymbol{\beta}$-sound [Hint: show that $\top \neq \omega$].

16.3.10. Prove that Theorem 16.1.10(ii) still holds if the condition of properness is replaced by the following two conditions

$$A \leq_{\mathcal{T}} B \;\Rightarrow\; C \to A \leq_{\mathcal{T}} C \to B$$

$$(A \to B) \cap (C \to D) \leq_{\mathcal{T}} A \cap C \to B \cap D.$$

16.3.11. Show that the following condition

$$A \to B =_{\mathcal{T}} \top \to \top \;\Rightarrow\; B =_{\mathcal{T}} \top$$

is necessary for the admissibility of rule ($\boldsymbol{\beta}$-*red*) in $\lambda_{\cap}^{\mathcal{T}}$. [Hint: Use Proposition 16.2.1(ii).]

16.3.12. Remember that the systems $\lambda_\cap^{\mathtt{K}}$ and $\lambda_{\cap^\top}^{\mathtt{K}^\top}$ are defined in Exercise 15.5.2.

(i) Show that rules ($\boldsymbol{\beta}$-*red*) and ($\boldsymbol{\beta}$I-*exp*) are admissible in $\lambda_\cap^{\mathtt{K}}$, while ($\boldsymbol{\beta}$-*exp*) is not admissible.

(ii) Show that rules ($\boldsymbol{\beta}$-*red*) and ($\boldsymbol{\beta}$-*exp*) are admissible in $\lambda_{\cap^\top}^{\mathtt{K}^\top}$.

16.3.13. (i) Show that for $\mathcal{T} \in \{\text{AO}, \text{Engeler}, \text{Plotkin}, \text{CDS}\}$ one has

$$\lambda_{\cap^\top}^{\mathcal{T}} \not\models (\boldsymbol{\eta}\text{-red}).$$

(ii) Show that for $\mathcal{T} = \text{CD}$ one has

$$\lambda_\cap^{\mathcal{T}} \not\models (\boldsymbol{\eta}\text{-red}).$$

16.3.14. Verify the following.

(i) $\boldsymbol{\eta}$-soundness implies $\boldsymbol{\eta}\top$-soundness.

(ii) Let $\mathcal{T} \in \{\text{BCD}, \text{Plotkin}, \text{Engeler}, \text{CDS}\}$. Then $\mathcal{T}$ is not $\boldsymbol{\eta}\top$-sound.

(iii) Let $\mathcal{T} \in \{\text{AO}, \text{CDV}, \text{CD}\}$. Then $\mathcal{T}$ is not $\boldsymbol{\eta}$-sound.

Notice that AO is $\boldsymbol{\eta}\top$-sound (Corollary 16.2.13). **Comment:** it is very interesting that AO is $\boldsymbol{\eta}\top$-sound but not $\boldsymbol{\eta}$-sound, why do you propose to erase it?

16.3.15. (i) Show that for $\mathcal{T} \in \{\text{BCD}, \text{Engeler}, \text{Plotkin}, \text{CDS}\}$ one has

$$\lambda_{\cap^\top}^{\mathcal{T}} \not\models (\boldsymbol{\eta}\text{-exp}).$$

(ii) Show that for $\mathcal{T} \in \{\text{CDV}, \text{CD}\}$ one has

$$\lambda_\cap^{\mathcal{T}} \not\models (\boldsymbol{\eta}\text{-exp}).$$

16.3.16. Show that rules ($\boldsymbol{\eta}$-*red*) and ($\boldsymbol{\eta}$-*exp*) are not admissible in the systems $\lambda_\cap^{\mathtt{K}}$ and $\lambda_{\cap^\top}^{\mathtt{K}^\top}$ as defined in Exercises 15.5.2.

16.3.17. Let $\vdash$ denote derivability in the system obtained from the system $\lambda_\cap^{\text{CDV}}$ by replacing rule ($\leq$) by the rules ($\cap$E), see Definition 15.2.5, and adding the rule

$$(R\eta) \quad \frac{\Gamma \vdash \lambda x.Mx : A}{\Gamma \vdash M : A} \quad \text{if } x \notin \text{FV}(M).$$

Show that $\Gamma \vdash_\cap^{\text{CDV}} M : A \ \Leftrightarrow\ \Gamma \vdash M : A$.

16.3.18. (Barendregt et al. [1983]) Let $\vdash$ denote derivability in the system obtained from $\lambda_{\cap^\top}^{\text{BCD}}$ by replacing rule ($\leq$) by the rules ($\cap$E) and adding ($R\eta$) as defined in Exercise 16.3.17. Verify that

$$\Gamma \vdash_{\cap^\top}^{\text{BCD}} M : A \ \Leftrightarrow\ \Gamma \vdash M : A.$$

16.3.19. Let Δ be a basis that is allowed to be infinite. We define $\Delta \vdash M : A$ iff there exists a finite basis $\Gamma \subseteq \Delta$ such that $\Gamma \vdash M : A$.

(i) Show that all the typability rules are derivable except possibly for ($\to$I).

(ii) Suppose dom(Δ) is the set of all the variables. Show that the rule ($\to$I) is derivable if it is reformulated as

$$\Delta_x, x{:}A \vdash M : B \;\Rightarrow\; \Delta \vdash (\lambda x.M) : (A \to B),$$

with Δ_x the result of removing any $x{:}C$ from Δ.

(iii) Reformulate and prove Propositions 15.2.8, 15.2.10, Theorems 16.1.1 and 16.1.10 for infinite bases.

16.3.20. A *multi-basis* Γ is a set of declarations, in which the requirement that

$$x{:}A, y{:}B \in \Gamma \;\Rightarrow\; x \equiv y \;\Rightarrow\; A \equiv B$$

is dropped. Let Δ be a (possibly infinite) multi-basis. We define $\Delta \vdash M : A$ iff there exists a singled (only one declaration per variable) basis $\Gamma \subseteq \Delta$ such that $\Gamma \vdash M : A$.

(i) Show that $x : \alpha_1, x : \alpha_2 \not\vdash^{\mathrm{CD}} x : \alpha_1 \cap \alpha_2$.

(ii) Show that $x : \alpha_1 \to \alpha_2, x : \alpha_1 \not\vdash^{\mathrm{CD}} xx : \alpha_2$.

(iii) Consider $\Delta = \{x : \alpha_1 \cap \alpha_2, x : \alpha_1\}$;
$A = \alpha_2$;
$B = (\alpha_1 \to \alpha_2 \to \alpha_3) \to \alpha_3$;
$M = \lambda y.yxx$.
Show that $\Delta, x : A \vdash^{\mathrm{CD}} M : B$, but $\Delta \not\vdash^{\mathrm{CD}} (\lambda x.M) : (A \to B)$.

(iv) We say that a multi-basis is closed under $\cap$ if for all $x \in \mathrm{dom}(\Delta)$ the set $\mathcal{X} = \Delta(x)$ is closed under $\cap$, i.e. $A, B \in \mathcal{X} \;\Rightarrow\; A \cap B \in \mathcal{X}$, up to equality of types in the TT under consideration.
Show that all the typability rules of Figures 15.4 and 15.6, except for ($\to$I), are derivable for (possibly infinite) multi-bases that are closed under $\cap$.

(v) Let Δ be closed under $\cap$. We define

$$\Delta[x := X] = \{y : \Delta(y) \mid y \neq x\} \cup \{x : A \mid A \in X\}.$$

Prove that the following reformulation of ($\to$I) using principal filters is derivable

$$\frac{\Delta[x :=\uparrow B] \vdash N : C}{\Delta \vdash \lambda x.N : B \to C}.$$

(vi) Prove Propositions 15.2.8, 15.2.10, Theorems 16.1.1 and 16.1.10 for (possible infinite) multi-bases reformulating the statements whenever it is necessary.

(vii) Prove that if $\Delta(x)$'s are filters then $\{A \mid \Delta \vdash x : A\} = \Delta(x)$.

16.3.21. Show that the inclusions suggested in 15.3 are strict.