Lambda Calculus

Intended meaning

The meaning of

$$\lambda x.3x$$

is the function

$$x \longmapsto 3x$$

that assigns to x the value $3x$ (3 times x)
So according to this intended meaning we have

$$(\lambda x.3x)(6) = 18.$$

The parentheses around the 6 are usually not written:

$$(\lambda x.3x)6 = 18$$

Principal axiom

$$(\lambda x.M)N =_\beta M[x := N]$$

Aim of λ-calculus: to capture the notion of human computability

Language

Alphabet

$$\Sigma = \{x, ', (,), \lambda, =\}$$

Language

$$\begin{array}{rcl} \texttt{variable} & := & \texttt{x} \mid \texttt{variable}' \\ \texttt{term} & := & \texttt{variable} \mid (\texttt{term}\ \texttt{term}) \mid (\lambda\, \texttt{variable}\ \texttt{term}) \\ \texttt{formula} & := & \texttt{term} = \texttt{term} \end{array}$$

Theory

Axioms

$$(\lambda x\, M)N = M[x{:} = N]$$

$$M = M$$

Rules

$$\begin{array}{rcl} M = N & \Rightarrow & N = M \\ M = N, N = L & \Rightarrow & M = N \\ M = N & \Rightarrow & ML = NL \\ M = N & \Rightarrow & LM = LN \\ M = N & \Rightarrow & \lambda x\, M = \lambda x\, N \end{array}$$

Bureaucracy

Substitution

M	$M[x:=N]$
x	N
y	y
PQ	$(P[x:=N])(Q[x:=N])$
$\lambda x\, P$	$\lambda x\, P$
$\lambda y\, P$	$\lambda y\,(P[x:=N])$

where $y \not\equiv x$

'Association to the left'

$$PQ_1 \ldots Q_n \equiv (..((PQ_1)Q_2) \ldots Q_n).$$

'Associating to the right'

$$\lambda x_1 \ldots x_n.M \equiv (\lambda x_1(\lambda x_2(..(\lambda x_n(M))..)))).$$

Outer parentheses are often omitted. For example

$$(\lambda x.x)y \equiv ((\lambda xx)y)$$

Examples

$$
\begin{array}{rclcrcl}
\mathsf{I} & \equiv & \lambda x.x & \Rightarrow & \mathsf{I}X & =_\beta & X \\
\mathsf{K} & \equiv & \lambda xy.x & \Rightarrow & \mathsf{K}XY & = & X \\
\mathsf{S} & \equiv & \lambda xyz.xz(yz) & \Rightarrow & \mathsf{S}XYZ & = & XZ(YZ) \\
\mathsf{D} & \equiv & \lambda x.xx & \Rightarrow & \mathsf{D}X & = & XX \\
\mathsf{B} & \equiv & \lambda xyz.x(yz) & \Rightarrow & \mathsf{B}XYZ & = & X(YZ)
\end{array}
$$

Set of lambda terms: Λ

Functions of two arguments can be simulated by unary functions

Let $f(x, y) = x^2 + y$

Define

$$
\begin{array}{rcll}
F_x(y) & = & x^2 + y, & \text{that is } F_x = \lambda y.x^2 + y \\
F(x) & = & F_x, & \text{that is } F = \lambda x.F_x
\end{array}
$$

Then $Fxy = F_x y = x^2 + y$. Thus $F = \lambda x.(\lambda y.x^2 + y)$

Fixed point theorem

THEOREM. For all $F \in \Lambda$ there is an $M \in \Lambda$ such that

$$FM =_\beta M$$

PROOF. Defines $\mathsf{W} \equiv \lambda x.F(xx)$ and $M \equiv \mathsf{WW}$. Then

$$\begin{array}{rcl} M & \equiv & \mathsf{WW} \\ & \equiv & (\lambda x.F(xx))\mathsf{W} \\ & = & F(\mathsf{WW}) \\ & \equiv & FM. \blacksquare \end{array}$$

COROLLARY. For any 'context' $C[\vec{x}, m]$ there exists a M such that

$$M\vec{X} = C[\vec{X}, M].$$

PROOF. M can be taken the fixed point of $\lambda m\vec{x}.C[\vec{x}, m]$.
Then $M\vec{X} = (\lambda m\vec{x}.C[\vec{x}, m])M\vec{X} = C[\vec{X}, M]$. $\blacksquare$

Consequences

We can construct terms Y, L, O, P such that

$$\begin{array}{rcll} \mathsf{Y}f & = & f(\mathsf{Y}f) & \text{producing fixed points;} \\ \mathsf{L} & = & \mathsf{LL} & \text{take } \mathsf{L} \equiv \mathsf{Y}\,\mathsf{D}; \\ \mathsf{O}x & = & \mathsf{O} & \text{take } \mathsf{O} \equiv \mathsf{Y}\,\mathsf{K}; \\ \mathsf{P} & = & \mathsf{P}x. & \end{array}$$

More Bureaucracy

$\lambda x.x$ and $\lambda y.y$ acting on M both give M

We write

$$\lambda x.x \equiv_\alpha \lambda y.y$$

"Names of *bound variables* may be changed".

NB

$$\begin{array}{rcl} \mathsf{K}MN & \equiv & (\lambda xy.x)MN \\ & \equiv & (((\lambda x(\lambda y\ x))M)N) \\ & = & ((\lambda yM)N) \\ & = & M \qquad\qquad \text{assuming that } y \text{ not in } M. \end{array}$$

But

$$\begin{array}{rcl} \mathsf{K}yz & \equiv & (((\lambda x(\lambda y\ x))y)z) \\ & =_? & ((\lambda y\ y)z) \\ & = & z?? \end{array} \quad \text{better:} \quad \begin{array}{rcl} \mathsf{K}yz & \equiv & (((\lambda x'(\lambda y'\ x'))y)z) \\ & = & (\lambda y'\ y)z \\ & = & y \qquad \text{as it should.} \end{array}$$