Chapter 4

Reduction

There is a certain asymmetry in the basic scheme (β). The statement

$$(\lambda x.x^2 + 1)3 = 10$$

can be interpreted as '10 is the result of computing $(\lambda x.x^2 + 1)3$', but not vice versa. This computational aspect will be expressed by writing

$$(\lambda x.x^2 + 1)3 \twoheadrightarrow 10$$

which reads '$(\lambda x.x^2 + 1)3$ *reduces to* 10'.

Apart from this conceptual aspect, reduction is also useful for an analysis of convertibility. The Church-Rosser theorem says that if two terms are convertible, then there is a term to which they both reduce. In many cases the inconvertibility of two terms can be proved by showing that they do not reduce to a common term.

4.1. Definition. (i) A binary relation R on Λ is called *compatible* (with the operations) if

$$\begin{aligned} M\,R\,N \quad \Rightarrow \quad & (ZM)\,R\,(ZN), \\ & (MZ)\,R\,(NZ) \text{ and} \\ & (\lambda x.M)\,R\,(\lambda x.N). \end{aligned}$$

(ii) A *congruence* relation on Λ is a compatible equivalence relation.

(iii) A *reduction* relation on Λ is a compatible, reflexive and transitive relation.

4.2. Definition. The binary relations $\rightarrow_\beta$, $\twoheadrightarrow_\beta$ and $=_\beta$ on Λ are defined inductively as follows.

(i) 1. $(\lambda x.M)N \rightarrow_\beta M[x := N]$;
   2. $M \rightarrow_\beta N \;\Rightarrow\; ZM \rightarrow_\beta ZN$, $MZ \rightarrow_\beta NZ$ and $\lambda x.M \rightarrow_\beta \lambda x.N$.

(ii) 1. $M \twoheadrightarrow_\beta M$;
   2. $M \rightarrow_\beta N \;\Rightarrow\; M \twoheadrightarrow_\beta N$;
   3. $M \twoheadrightarrow_\beta N, N \twoheadrightarrow_\beta L \;\Rightarrow\; M \twoheadrightarrow_\beta L$.

(iii) 1. $M \twoheadrightarrow_\beta N \;\Rightarrow\; M =_\beta N$;
2. $M =_\beta N \;\Rightarrow\; N =_\beta M$;
3. $M =_\beta N, N =_\beta L \;\Rightarrow\; M =_\beta$L.

These relations are pronounced as follows.

$$\begin{array}{rcl} M \twoheadrightarrow_\beta N & : & M\beta\text{-}\textit{reduces} \text{ to } N; \\ M \to_\beta N & : & M\beta\text{-}\textit{reduces} \text{ to } N \textit{in one step}; \\ M =_\beta N & : & M \text{ is } \beta\text{-}\textit{convertible} \text{ to } N. \end{array}$$

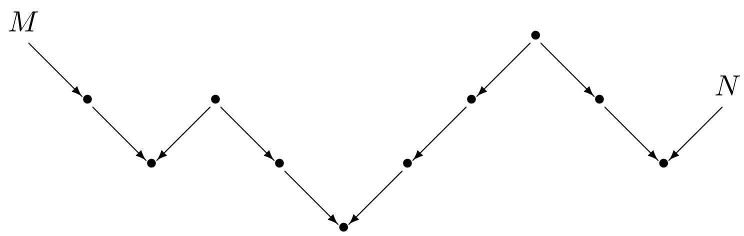
By definition $\to_\beta$ is compatible, $\twoheadrightarrow_\beta$ is a reduction relation and $=_\beta$ is a congruence relation.

4.3. Example. (i) Define

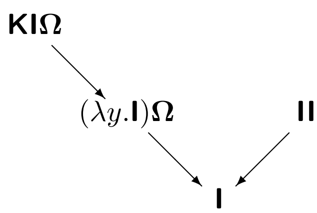
$$\begin{array}{rcl} \boldsymbol{\omega} & \equiv & \lambda x.xx, \\ \boldsymbol{\Omega} & \equiv & \boldsymbol{\omega\omega}. \end{array}$$

Then $\boldsymbol{\Omega} \to_\beta \boldsymbol{\Omega}$.
(ii) $\mathsf{KI}\boldsymbol{\Omega} \twoheadrightarrow_\beta \mathsf{I}$.

Intuitively, $M =_\beta N$ if M is connected to N via $\to_\beta$-arrows (disregarding the directions of these). In a picture this looks as follows.



4.4. Example. $\mathsf{KI}\boldsymbol{\Omega} =_\beta \mathsf{II}$. This is demonstrated by the following reductions.



4.5. Proposition. $M =_\beta N \;\Leftrightarrow\; \boldsymbol{\lambda} \vdash M = N.$

Proof. By an easy induction. □

4.6. Definition. (i) A *β-redex* is a term of the form $(\lambda x.M)N$. In this case $M[x := N]$ is its *contractum*.
(ii) A λ-term M is a *β-normal form* (β-nf) if it does not have a β-redex as subexpression.
(iii) A term M *has* a β-normal form if $M =_\beta N$ and N is a β-nf, for some N.

4.7. EXAMPLE. $(\lambda x.xx)y$ is not a β-nf, but has as β-nf the term yy.

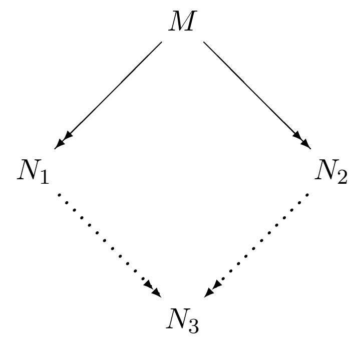
An immediate property of nf's is the following.

4.8. LEMMA. *Let M be a β-nf. Then*

$$M \twoheadrightarrow_\beta N \;\Rightarrow\; N \equiv M.$$

PROOF. This is true if $\twoheadrightarrow_\beta$ is replaced by $\rightarrow_\beta$. Then the result follows by transitivity. □

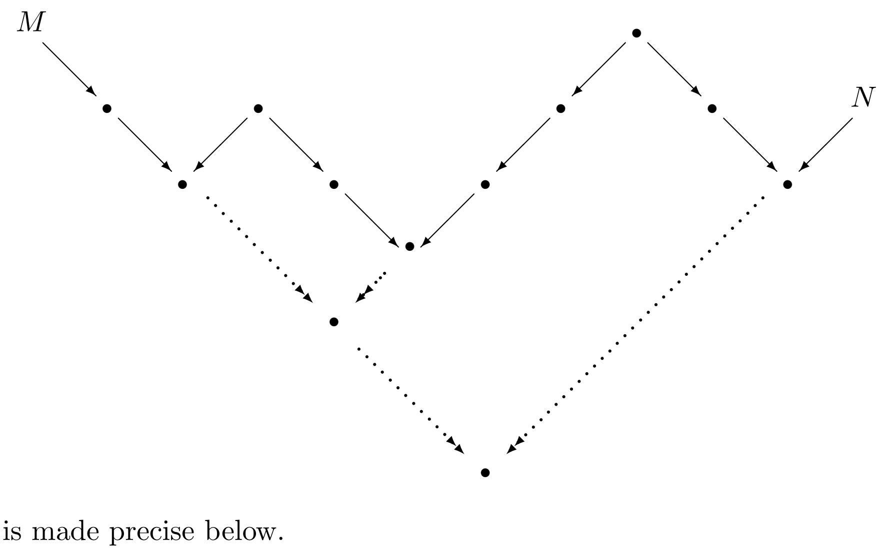
4.9. CHURCH-ROSSER THEOREM. *If $M \twoheadrightarrow_\beta N_1$, $M \twoheadrightarrow_\beta N_2$, then for some N_3 one has $N_1 \twoheadrightarrow_\beta N_3$ and $N_2 \twoheadrightarrow_\beta N_3$; in diagram*



The proof is postponed until 4.19.

4.10. COROLLARY. *If $M =_\beta N$, then there is an L such that $M \twoheadrightarrow_\beta L$ and $N \twoheadrightarrow_\beta L$.*

An intuitive proof of this fact proceeds by a tiling procedure: given an arrow path showing $M =_\beta N$, apply the Church-Rosser property repeatedly in order to find a common reduct. For the example given above this looks as follows.



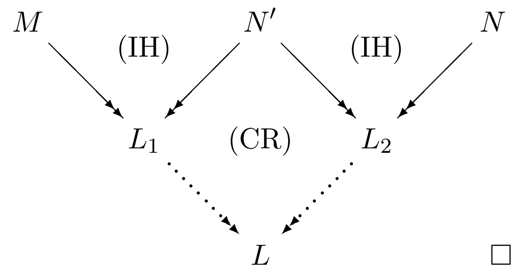
This is made precise below.

PROOF. Induction on the generation of $=_\beta$.

Case 1. $M =_\beta N$ because $M \twoheadrightarrow_\beta N$. Take $L \equiv N$.

Case 2. $M =_\beta N$ because $N =_\beta M$. By the IH there is a common β-reduct L_1 of N, M. Take $L \equiv L_1$.

Case 3. $M =_\beta N$ because $M =_\beta N'$, $N' =_\beta N$. Then



□

4.11. COROLLARY. (i) *If M has N as β-nf, then $M \twoheadrightarrow_\beta N$.*

(ii) *A λ-term has at most one β-nf.*

PROOF. (i) Suppose $M =_\beta N$ with N in β-nf. By Corollary 4.10 $M \twoheadrightarrow_\beta L$ and $N \twoheadrightarrow_\beta L$ for some L. But then $N \equiv L$, by Lemma 4.8, so $M \twoheadrightarrow_\beta N$.

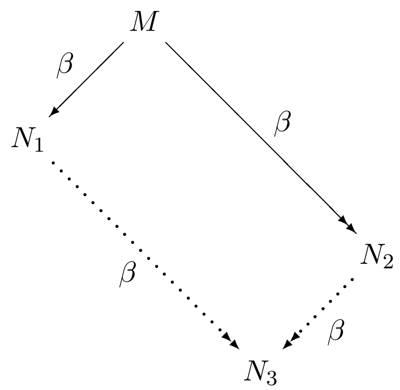
(ii) Suppose M has β-nf's N_1, N_2. Then $N_1 =_\beta N_2$ $(=_\beta M)$. By Corollary 4.10 $N_1 \twoheadrightarrow_\beta L$, $N_2 \twoheadrightarrow_\beta L$ for some L. But then $N_1 \equiv L \equiv N_2$ by Lemma 4.8. □

4.12. SOME CONSEQUENCES. (i) The λ-calculus is consistent, i.e. $\boldsymbol{\lambda} \nvdash$ **true** $=$ **false**. Otherwise **true** $=_\beta$ **false** by Proposition 4.5, which is impossible by Corollary 4.11 since **true** and **false** are distinct β-nf's. This is a syntactic consistency proof.

(ii) $\boldsymbol{\Omega}$ has no β-nf. Otherwise $\boldsymbol{\Omega} \twoheadrightarrow_\beta N$ with N in β-nf. But $\boldsymbol{\Omega}$ only reduces to itself and is not in β-nf.

(iii) In order to find the β-nf of a term M (if it exists), the various subexpressions of M may be reduced in different orders. By Corollary 4.11 (ii) the β-nf is unique.

The proof of the Church-Rosser theorem occupies 4.13–4.19. The idea of the proof is as follows. In order to prove Theorem 4.9, it is sufficient to show the Strip Lemma:



In order to prove this lemma, let $M \to_\beta N_1$ be a one step reduction resulting from changing a redex R in M in its contractum R' in N_1. If one makes a bookkeeping of what happens with R during the reduction $M \twoheadrightarrow_\beta N_2$, then by reducing all 'residuals' of R in N_2 the term N_3 can be found. In order to do the necessary bookkeeping an extended set $\underline{\Lambda} \supseteq \Lambda$ and reduction $\underline{\beta}$ is introduced. The underlining serves as a 'tracing isotope'.

4.13. Definition (Underlining). (i) $\underline{\Lambda}$ is the set of terms defined inductively as follows.

$$\begin{array}{rcl} x \in V & \Rightarrow & x \in \underline{\Lambda}, \\ M, N \in \underline{\Lambda} & \Rightarrow & (MN) \in \underline{\Lambda}, \\ M \in \underline{\Lambda}, x \in V & \Rightarrow & (\lambda x.M) \in \underline{\Lambda}, \\ M, N \in \underline{\Lambda}, x \in V & \Rightarrow & ((\underline{\lambda} x.M)N) \in \underline{\Lambda}. \end{array}$$

(ii) The underlined reduction relations $\rightarrow_{\underline{\beta}}$ (one step) and $\twoheadrightarrow_{\underline{\beta}}$ are defined starting with the contraction rules

$$\begin{array}{rcl} (\lambda x.M)N & \rightarrow_{\underline{\beta}} & M[x := N], \\ (\underline{\lambda} x.M)N & \rightarrow_{\underline{\beta}} & M[x := N]. \end{array}$$

Then $\rightarrow_{\underline{\beta}}$ is extended in order to become a compatible relation (also with respect to $\underline{\lambda}$-abstraction). Moreover, $\twoheadrightarrow_{\underline{\beta}}$ is the transitive reflexive closure of $\rightarrow_{\underline{\beta}}$.

(iii) If $M \in \underline{\Lambda}$, then $|M| \in \Lambda$ is obtained from M by leaving out all underlinings. E.g. $|(\lambda x.x)((\underline{\lambda} x.x)(\lambda x.x))| \equiv \mathbf{I}(\mathbf{II})$.

4.14. Definition. The map $\varphi : \underline{\Lambda} \rightarrow \Lambda$ is defined inductively as follows.

$$\begin{array}{rcl} \varphi(x) & \equiv & x, \\ \varphi(MN) & \equiv & \varphi(M)\varphi(N), \\ \varphi(\lambda x.M) & \equiv & \lambda x.\varphi(M), \\ \varphi((\underline{\lambda} x.M)N) & \equiv & \varphi(M)[x := \varphi(N)]. \end{array}$$

In other words, φ contracts all redexes that are underlined, from the inside to the outside.

Notation. If $|M| \equiv N$ or $\varphi(M) \equiv N$, then this will be denoted by

$$M \xrightarrow[|\,|]{} N \text{ or } M \xrightarrow[\varphi]{} N.$$

4.15. Lemma.

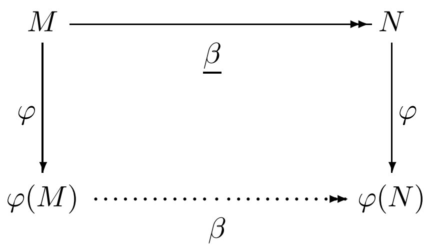
$$\begin{array}{ccc} M' & \overset{\cdots\cdots\cdots\cdots}{\underset{\underline{\beta}}{\twoheadrightarrow}} & N' \\ {\scriptstyle |\,|}\downarrow & & \downarrow{\scriptstyle |\,|} \\ M & \underset{\beta}{\twoheadrightarrow} & N \end{array} \qquad \begin{array}{l} M', N' \in \underline{\Lambda}, \\ M, N \in \Lambda. \end{array}$$

PROOF. First suppose $M \to_\beta N$. Then N is obtained by contracting a redex in M and N' can be obtained by contracting the corresponding redex in M'. The general statement follows by transitivity. □

4.16. LEMMA. (i) *Let* $M, N \in \underline{\Lambda}$. *Then*

$$\varphi(M[x := N]) \equiv \varphi(M)[x := \varphi(N)].$$

(ii)

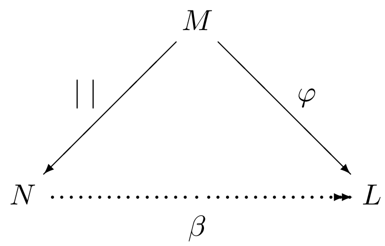


$M, N \in \underline{\Lambda}$.

PROOF. (i) By induction on the structure of M, using the Substitution Lemma (see Exercise 2.2) in case $M \equiv (\underline{\lambda} y.P)Q$. The condition of that lemma may be assumed to hold by our convention about free variables.

(ii) By induction on the generation of $\twoheadrightarrow_{\underline{\beta}}$, using (i). □

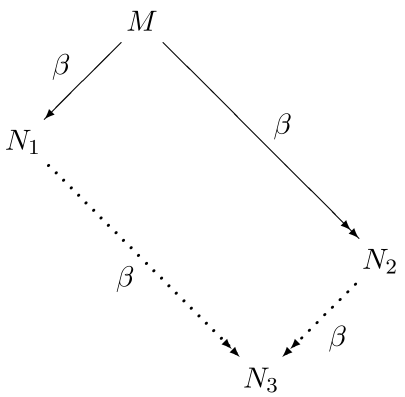
4.17. LEMMA.



$M \in \underline{\Lambda}$,
$N, L \in \Lambda$.

PROOF. By induction on the structure of M. □

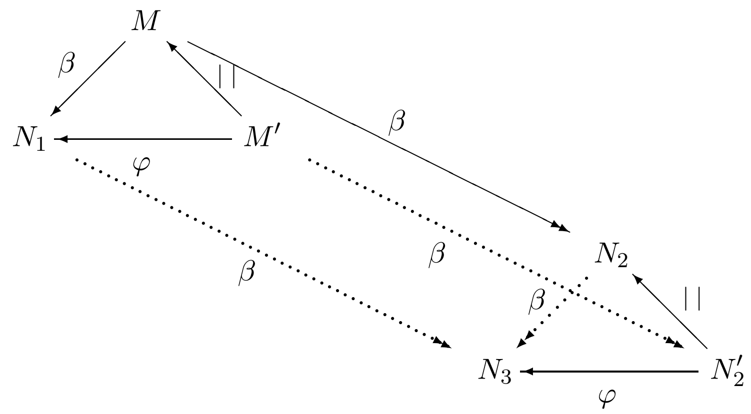
4.18. STRIP LEMMA.



$M, N_1, N_2, N_3 \in \Lambda$.

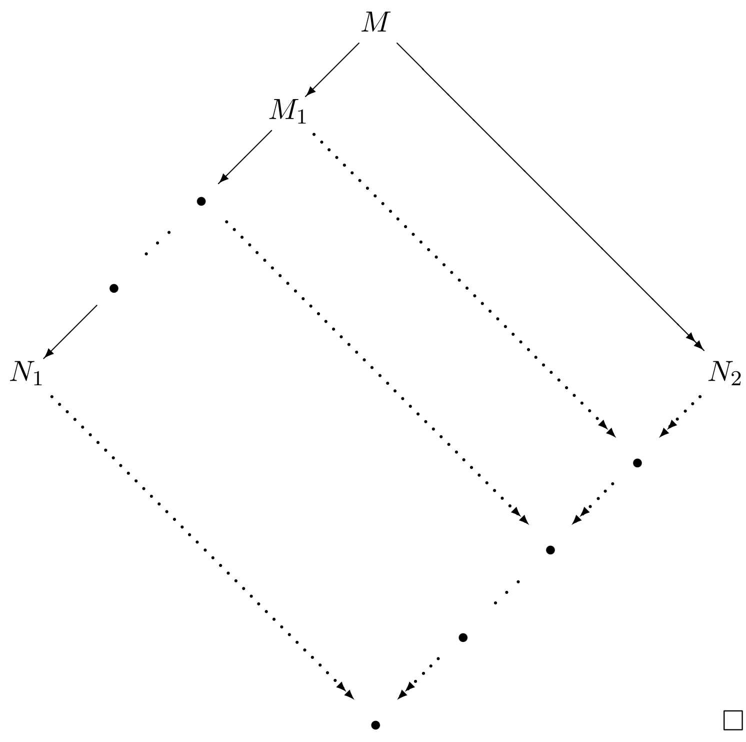
PROOF. Let N_1 be the result of contracting the redex occurrence $R \equiv (\lambda x.P)Q$ in M. Let $M' \in \underline{\Lambda}$ be obtained from M by replacing R by $R' \equiv (\underline{\lambda} x.P)Q$. Then

$|M'| \equiv M$ and $\varphi(M') \equiv N_1$. By the lemmas 4.15, 4.16 and 4.17 we can erect the diagram



which proves the Strip Lemma. □

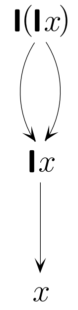
4.19. Proof of the Church-Rosser Theorem. If $M \twoheadrightarrow_\beta N_1$, then $M \equiv M_1 \to_\beta M_2 \to_\beta \cdots \to_\beta M_n \equiv N_1$. Hence the CR property follows from the Strip Lemma and a simple diagram chase:



□

4.20. Definition. For $M \in \Lambda$ the *reduction graph* of M, notation $G_\beta(M)$, is the directed multigraph with vertices $\{N \mid M \twoheadrightarrow_\beta N\}$ and directed by $\to_\beta$.

4.21. EXAMPLE. $G_\beta(\mathbf{I}(\mathbf{I}x))$ is



sometimes simply drawn as

It can happen that a term M has a nf, but at the same time an infinite reduction path. Let $\boldsymbol{\Omega} \equiv (\lambda x.xx)(\lambda x.xx)$. Then $\boldsymbol{\Omega} \to \boldsymbol{\Omega} \to \cdots$ so $\mathbf{KI}\boldsymbol{\Omega} \to \mathbf{KI}\boldsymbol{\Omega} \to \cdots$, and $\mathbf{KI}\boldsymbol{\Omega} \twoheadrightarrow \mathbf{I}$. Therefore a so called *strategy* is necessary in order to find the normal form. We state the following theorem; for a proof see Barendregt (1984), Theorem 13.2.2.

4.22. NORMALIZATION THEOREM. *If M has a normal form, then iterated contraction of the leftmost redex leads to that normal form.*

In other words: the leftmost reduction strategy is *normalizing*. This fact can be used to find the normal form of a term, or to prove that a certain term has no normal form.

4.23. EXAMPLE. $\mathbf{K}\boldsymbol{\Omega}\mathbf{I}$ has an infinite leftmost reduction path, viz.

$$\mathbf{K}\boldsymbol{\Omega}\mathbf{I} \to_\beta (\lambda y.\boldsymbol{\Omega})\mathbf{I} \to_\beta \boldsymbol{\Omega} \to_\beta \boldsymbol{\Omega} \to_\beta \cdots,$$

and hence does not have a normal form.

The functional language (pure) *Lisp* uses an *eager* or *applicative* evaluation strategy, i.e. whenever an expression of the form FA has to be evaluated, A is reduced to normal form first, before 'calling' F. In the λ-calculus this strategy is not normalizing as is shown by the two reduction paths for $\mathbf{KI}\boldsymbol{\Omega}$ above. There is, however, a variant of the lambda calculus, called the λI-calculus, in which the eager evaluation strategy is normalizing. In this λI-calculus terms like $\mathbf{K}$, 'throwing away' $\boldsymbol{\Omega}$ in the reduction $\mathbf{KI}\boldsymbol{\Omega} \twoheadrightarrow \mathbf{I}$ do not exist. The 'ordinary' λ-calculus is sometimes referred to as λK-calculus; see Barendregt (1984), Chapter 9.

Remember the fixedpoint combinator $\mathbf{Y}$. For each $F \in \Lambda$ one has $\mathbf{Y}F =_\beta F(\mathbf{Y}F)$, but neither $\mathbf{Y}F \twoheadrightarrow_\beta F(\mathbf{Y}F)$ nor $F(\mathbf{Y}F) \twoheadrightarrow_\beta \mathbf{Y}F$. In order to solve *reduction* equations one can work with A.M. Turing's fixedpoint combinator, which has a different reduction behaviour.

4.24. Definition. Turing's fixedpoint combinator $\mathbf{\Theta}$ is defined by setting

$$\begin{aligned} A &\equiv \lambda xy.y(xxy), \\ \mathbf{\Theta} &\equiv AA. \end{aligned}$$

4.25. Proposition. *For all $F \in \Lambda$ one has*

$$\mathbf{\Theta} F \twoheadrightarrow_\beta F(\mathbf{\Theta} F).$$

Proof.

$$\begin{aligned} \mathbf{\Theta} F &\equiv AAF \\ &\rightarrow_\beta (\lambda y.y(AAy))F \\ &\rightarrow_\beta F(AAF) \\ &\equiv F(\mathbf{\Theta} F). \ \square \end{aligned}$$

4.26. Example. $\exists G\, \forall X\; GX \twoheadrightarrow X(XG)$. Indeed,

$$\begin{aligned} \forall X\; GX \twoheadrightarrow X(XG) &\Leftarrow G \twoheadrightarrow \lambda x.x(xG) \\ &\Leftarrow G \twoheadrightarrow (\lambda gx.x(xg))G \\ &\Leftarrow G \equiv \mathbf{\Theta}(\lambda gx.x(xg)). \end{aligned}$$

Also the Multiple Fixedpoint Theorem has a 'reducing' variant.

4.27. Theorem. *Let $F_1, \ldots, F_n$ be λ-terms. Then we can find $X_1, \ldots, X_n$ such that*

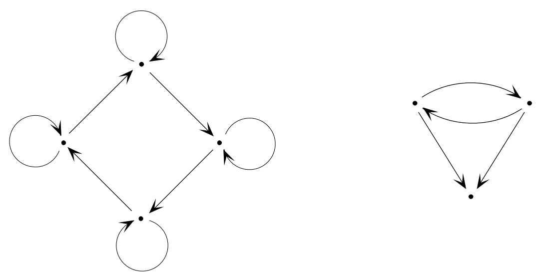
$$\begin{aligned} X_1 &\twoheadrightarrow F_1 X_1 \cdots X_n, \\ &\vdots \\ X_n &\twoheadrightarrow F_n X_1 \cdots X_n. \end{aligned}$$

Proof. As for the equational Multiple Fixedpoint Theorem 3.17, but now using $\mathbf{\Theta}$. $\square$

Exercises

4.1. Show $\forall M\, \exists N\, [N$ in β-nf and $N\mathsf{I} \twoheadrightarrow_\beta M]$.

4.2. Construct four terms M with $G_\beta(M)$ respectively as follows.



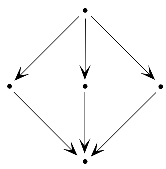
4.3. Show that there is no $F \in \Lambda$ such that for all $M, N \in \Lambda$

$$F(MN) = M.$$

4.4.* Let $M \equiv AAx$ with $A \equiv \lambda axz.z(aax)$. Show that $G_\beta(M)$ contains as subgraphs an n-dimensional cube for every $n \in \mathbb{N}$.

4.5. (A. Visser)
(i) Show that there is only one redex R such that $G_\beta(R)$ is as follows.

(ii) Show that there is no $M \in \Lambda$ with $G_\beta(M)$ is



[*Hint.* Consider the relative positions of redexes.]

4.6.* (C. Böhm) Examine $G_\beta(M)$ with M equal to
(i) $H\mathbf{I}H, \quad H \equiv \lambda xy.x(\lambda z.yzy)x.$
(ii) $LL\mathbf{I}, \quad L \equiv \lambda xy.x(yy)x.$
(iii) $Q\mathbf{I}Q, \quad Q \equiv \lambda xy.xy\mathbf{I}xy.$

4.7.* (J.W. Klop) Extend the λ-calculus with two constants $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$. The reduction rules are extended to include $\boldsymbol{\delta}MM \to \boldsymbol{\varepsilon}$. Show that the resulting system is not Church-Rosser.
[*Hint.* Define terms C, D such that

$$\begin{array}{rcl} Cx & \twoheadrightarrow & \boldsymbol{\delta}x(Cx) \\ D & \twoheadrightarrow & CD \end{array}$$

Then $D \twoheadrightarrow \boldsymbol{\varepsilon}$ and $D \twoheadrightarrow C\boldsymbol{\varepsilon}$ in the extended reduction system, but there is no common reduct.]

4.8. Show that the term $M \equiv AAx$ with $A \equiv \lambda axz.z(aax)$ does not have a normal form.

4.9. (i) Show $\boldsymbol{\lambda} \nvdash WWW = \boldsymbol{\omega}_3\boldsymbol{\omega}_3$, with $W \equiv \lambda xy.xyy$ and $\boldsymbol{\omega}_3 \equiv \lambda x.xxx$.
(ii) Show $\boldsymbol{\lambda} \nvdash B_x = B_y$ with $B_z \equiv A_zA_z$ and $A_z \equiv \lambda p.ppz$.

4.10. Draw $G_\beta(M)$ for M equal to:
(i) $WWW, \quad W \equiv \lambda xy.xyy.$
(ii) $\boldsymbol{\omega}\boldsymbol{\omega}, \quad \boldsymbol{\omega} \equiv \lambda x.xx.$
(iii) $\boldsymbol{\omega}_3\boldsymbol{\omega}_3, \quad \boldsymbol{\omega}_3 \equiv \lambda x.xxx.$
(iv) $(\lambda x.\mathbf{I}xx)(\lambda x.\mathbf{I}xx).$
(v) $(\lambda x.\mathbf{I}(xx))(\lambda x.\mathbf{I}(xx)).$
(vi) $\mathbf{I}\mathbf{I}(\mathbf{I}\mathbf{I}\mathbf{I}).$

4.11. The *length* of a term is its number of symbols times 0.5 cm. Write down a λ-term of length < 30 cm with normal form $> 10^{10^{10}}$ light year.
[*Hint.* Use Proposition 2.15 (ii). The speed of light is $c = 3 \times 10^{10}$ cm/s.]