Completeness of two systems of illative combinatory logic for first-order propositional and predicate calculus[1]

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Abstract

Illative combinatory logic consists of the theory of combinators or lambda calculus extended by extra constants (and corresponding axioms and rules) intended to capture inference. The paper considers systems of illative combinatory logic that are sound for first order propositional and predicate calculus. The interpretation from ordinary logic into the illative systems can be done in two ways: following the propositions-as-types paradigm, in which derivations become combinators, or in a more direct way, in which derivations are not translated. Both translations are closely related in a canonical way. In a preceding paper, BBD[1993], we proved completeness of the two direct translations. In the present paper we prove that also the two indirect translations are complete. These papers fulfill the program of Church and Curry to base logic on a consistent system of λ-terms or combinators. Hitherto this program had failed because systems of ICL were either too weak (to provide a sound interpretation) or too strong (sometimes even inconsistent).

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1. Introduction

The theory of combinators and the lambda calculus are theories that succesfully analyze the notion of effective computability. However, the original founders of these subjects, Curry and Church, also had as aim to provide a basis for logic (and thereby mathematics). Unfortunately, it was shown by Kleene and Rosser that their systems were inconsistent. Curry and his school then started a program of defining several systems of illative combinatory logic (ICL) of varying strength. The goal was to find stronger and stronger systems which were consistent and weaker and weaker systems which were inconsistent but strong enough to interpret logic, hoping to end up with a consistent system in which logic could be interpreted.

Following this methodology, Bunder introduced restrictions on the rules of the illative constants so that first order propositional and predicate calculus could be interpreted in the resulting systems. In all his systems the usual derivation of Curry's paradox is blocked, but the consistency of the systems remains an open question. In BBD[1993] we introduced 4 systems of illative combinatory logic. We derived roughly the following soundness result

$$\Delta \vdash_{\mathrm{L}} A \;\Rightarrow\; [\Delta] \vdash_{\mathrm{C}} [A],$$

where L represents propositional or predicate logic and $[-]$ one of two possible translations of each system into an ICL system C. Of the interpretations one is the propositions-as-types interpretation due to Curry, Howard and de Bruijn, the other is a more direct interpretation.

We derived completeness results for the direct translations of propositional and predicate calculus into 2 of the 4 systems of ICL. These, again roughly, took the following form

$$[\Delta] \vdash_{\mathrm{C}} [A] \;\Rightarrow\; \Delta \vdash_{\mathrm{L}} A.$$

In the present paper we shall prove that also the two indirect translations are complete.
These completeness results imply the consistency of the ICL's involved.

2. Some definitions of preceding paper

This paper is a continuation of a preceding paper, BBD[1993]. We will refer to that paper most of the time simply as B; so for example Definition B2.1 means Definition 2.1 in BBD[1993]. In this section we repeat some definitions of BBD[1993] that are needed in the present paper. For an introduction into ICL-systems, motivations, examples, propositions with proofs, and references we refer to BBD[1993].

2.1. DEFINITION (Definition B2.1; the logic system PROP). PROP is the $\supset$ fragment of intuitionistic propositional logic determined as follows.

(i) The set of formulas of PROP, notation $\mathbb{F}_{\mathrm{PROP}}$, is defined by the following abstract syntax

$$\mathbb{F}_{\mathrm{PROP}} = \mathbb{V} \mid \mathbb{F}_{\mathrm{PROP}} \supset \mathbb{F}_{\mathrm{PROP}}.$$

Here $\mathbb{V}$ is a set of propositional variables.

(ii) Let $\Gamma \subseteq \mathbb{F}_{\mathrm{PROP}}$ and $\varphi \in \mathbb{F}_{\mathrm{PROP}}$. Then $\Gamma \vdash_{\mathrm{PROP}} \varphi$ is defined by the following system of natural deduction.

PROP

$$\begin{array}{rcl} \varphi\in\Gamma & \Rightarrow & \Gamma \vdash \varphi; \\ \Gamma \vdash \varphi \supset \psi,\ \Gamma \vdash \varphi & \Rightarrow & \Gamma \vdash \psi; \\ \Gamma, \varphi \vdash \psi & \Rightarrow & \Gamma \vdash \varphi \supset \psi. \end{array}$$

2.2. DEFINITION (Definition B2.2; the logic system PRED). PRED is the $\{\supset, \forall\}$ fragment of first order many-sorted intuitionistic predicate calculus of a given signature s.

Below as an example, we will treat a version of PRED with s the signature of the structure

$$< \mathbf{A}_1, \mathbf{A}_2, \mathbf{f}, \mathbf{g}, \mathbf{P}, \mathbf{a} >$$

with

$$\begin{array}{rl} \mathbf{A}_1, \mathbf{A}_2 & \text{nonempty sets;} \\ \mathbf{f} : \mathbf{A}_1 \to \mathbf{A}_1 & \text{a unary function;} \\ \mathbf{g} : \mathbf{A}_1 \to \mathbf{A}_2 \to \mathbf{A}_1 & \text{a binary function;} \\ \mathbf{P} \subseteq \mathbf{A}_1 & \text{a unary relation;} \\ \mathbf{a} \in \mathbf{A}_1 & \text{a constant.} \end{array}$$

(All results also hold for arbitrary signatures.)

(i) The set of terms of PRED, notation $\mathbb{T}_{\mathrm{PRED}}$, is defined by the following abstract syntax.

$$\begin{array}{rcl} \mathbb{T}_{\mathrm{PRED}} & = & \mathbb{T}_{\mathbf{A}_1} \mid \mathbb{T}_{\mathbf{A}_2} \\ \mathbb{T}_{\mathbf{A}_1} & = & \mathbb{V}^{\mathbf{A}_1} \mid \mathbf{a} \mid \mathbf{f}\mathbb{T}_{\mathbf{A}_1} \mid \mathbf{g}\mathbb{T}_{\mathbf{A}_1}\mathbb{T}_{\mathbf{A}_2} \\ \mathbb{T}_{\mathbf{A}_2} & = & \mathbb{V}^{\mathbf{A}_2}. \end{array}$$

(ii) The set of formulas of PRED, notation $\mathbb{F}_{\mathrm{PRED}}$, is defined by the following abstract syntax.

$$\mathbb{F}_{\mathrm{PRED}} = \mathbf{P}\mathbb{T}_{\mathbf{A}_1} \mid \mathbb{F}_{\mathrm{PRED}} \supset \mathbb{F}_{\mathrm{PRED}} \mid \forall \mathbb{V}^{\mathbf{A}_i} \mathbb{F}_{\mathrm{PRED}}$$

(iii) $\Gamma \vdash_{\mathrm{PRED}} \varphi$ is axiomatised by the following system of natural deduction.

PRED

$$\begin{array}{rcl}
\varphi\in\Gamma & \Rightarrow & \Gamma \vdash \varphi; \\
\Gamma \vdash \varphi \supset \psi,\ \Gamma \vdash \varphi & \Rightarrow & \Gamma \vdash \psi; \\
\Gamma, \varphi \vdash \psi & \Rightarrow & \Gamma \vdash \varphi \supset \psi; \\
\Gamma \vdash \forall x^{\mathbf{A}_i}\varphi,\ t\in\mathbb{T}_{\mathbf{A}_i} & \Rightarrow & \Gamma \vdash \varphi[x^{\mathbf{A}_i} := t]; \\
\Gamma \vdash \varphi,\ x^{\mathbf{A}_i}\notin \mathrm{FV}(\Gamma) & \Rightarrow & \Gamma \vdash \forall x^{\mathbf{A}_i}\varphi.
\end{array}$$

The systems PROP and PRED will be interpreted in ICL's by two kinds of embeddings. The first kind is 'direct' and the second kind is according to the 'propositions-as-types' and 'proofs-as-terms' paradigm.. As there are two logical systems, PROP and PRED, there will be four systems of ICL. These systems are called $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathbf{F}$, and $\mathcal{I}\mathbf{G}$ respectively. Their use for the two kinds of interpretation is as follows. Let $[\ ]^1$ be the direct, and $[\ ]^2$ the propositions-as-types translation. Then the following table shows the systems of ICL that are used for the two translations of PROP and PRED.

	$[\ ]^1$	$[\ ]^2$
PROP	$\mathcal{I}\mathbf{P}$	$\mathcal{I}\mathbf{F}$
PRED	$\mathcal{I}\mathbf{\Xi}$	$\mathcal{I}\mathbf{G}$

For example

$$[\ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathbf{G}.$$

Now we will describe the four systems of ICL.

2.3. Definition (Definition B2.3; the four systems of ICL). Let $\mathrm{T} = \Lambda(\mathbf{\Xi}, \mathbf{L})$, be the set of type-free lambda terms extended by the extra constants $\mathbf{\Xi}$ and $\mathbf{L}$.

(i) Define the following terms in T.

$$\begin{array}{rcl}
\mathbf{P} & \equiv & \lambda xy.\mathbf{\Xi}(\mathbf{K}x)(\mathbf{K}y), \\
\mathbf{F} & \equiv & \lambda xyz.\mathbf{\Xi}x(y\circ z), \\
\mathbf{G} & \equiv & \lambda xyz.\mathbf{\Xi}x(\mathbf{S}yz), \\
\mathbf{H} & \equiv & \mathbf{L}\circ\mathbf{K},
\end{array}$$

where $\mathsf{K} \equiv \lambda pq.p$, $M \circ N \equiv \lambda x.M(Nx)$ and $\mathsf{S} \equiv \lambda pqr.pr(qr)$.
Write $X \supset Y$ for $\mathsf{P}XY$.

(ii) Define the following four systems of illative combinatory logic $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{\Xi}$, $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$. T is the set of terms of all 4 systems. A basis Γ is a set of terms. All four systems have as rules the following:

All systems

$$\begin{array}{rcl} X \in \Gamma & \Rightarrow & \Gamma \vdash X; \\ \Gamma \vdash X, X =_{\beta\eta} Y & \Rightarrow & \Gamma \vdash Y. \end{array}$$

The four systems have the following specific rules:

$\mathcal{I}\mathsf{P}$

$$\begin{array}{lrcl} \mathsf{P}_\mathrm{e} & \Gamma \vdash X \supset Y,\ \Gamma \vdash X & \Rightarrow & \Gamma \vdash Y; \\ \mathsf{P}_\mathrm{i} & \Gamma, X \vdash Y,\ \Gamma \vdash \mathsf{H}X & \Rightarrow & \Gamma \vdash X \supset Y; \\ \mathsf{P}_\mathsf{H} & \Gamma, X \vdash \mathsf{H}Y,\ \Gamma \vdash \mathsf{H}X & \Rightarrow & \Gamma \vdash \mathsf{H}(X \supset Y). \end{array}$$

$\mathcal{I}\mathsf{\Xi}$

$$\begin{array}{lrcl} \Xi_\mathrm{e} & \Gamma \vdash \mathsf{\Xi}XY,\ \Gamma \vdash XV & \Rightarrow & \Gamma \vdash YV; \\ \Xi_\mathrm{i} & \Gamma, Xx \vdash Yx,\ \Gamma \vdash \mathsf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y) & \Rightarrow & \Gamma \vdash \mathsf{\Xi}XY; \\ \Xi_\mathsf{H} & \Gamma, Xx \vdash \mathsf{H}(Yx),\ \Gamma \vdash \mathsf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y) & \Rightarrow & \Gamma \vdash \mathsf{H}(\mathsf{\Xi}XY). \end{array}$$

$\mathcal{I}\mathsf{F}$

$$\begin{array}{lrcl} \mathsf{F}_\mathrm{e} & \Gamma \vdash \mathsf{F}XYZ,\ \Gamma \vdash XV & \Rightarrow & \Gamma \vdash Y(ZV); \\ \mathsf{F}_\mathrm{i} & \Gamma, Xx \vdash Y(Zx),\ \Gamma \vdash \mathsf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y, Z) & \Rightarrow & \Gamma \vdash \mathsf{F}XYZ; \\ \mathsf{F}_\mathsf{L} & \Gamma, Xx \vdash \mathsf{L}Y,\ \Gamma \vdash \mathsf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y) & \Rightarrow & \Gamma \vdash \mathsf{L}(\mathsf{F}XY). \end{array}$$

$\mathcal{I}\mathsf{G}$

$\mathsf{G_e}$	$\Gamma \vdash \mathbf{G}XYZ,\ \ \Gamma \vdash XV$	$\Rightarrow$	$\Gamma \vdash YV(ZV);$
$\mathsf{G_i}$	$\Gamma, Xx \vdash Yx(Zx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y, Z)$	$\Rightarrow$	$\Gamma \vdash \mathbf{G}XYZ;$
$\mathsf{G_L}$	$\Gamma, Xx \vdash \mathbf{L}(Yx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y)$	$\Rightarrow$	$\Gamma \vdash \mathbf{L}(\mathbf{G}XY).$

2.4. Definition (Definition B2.8; interpretation of PROP). Let r be a closed term in $\Lambda(\mathbf{\Xi}, \mathbf{L})$. Two maps (for $i = 1, 2$)

$$[-]^i_r : \mathbb{F}_{\mathrm{PROP}} \to \Lambda(\mathbf{\Xi}, \mathbf{L})$$

and two maps

$$\Gamma^i_r : \mathbb{F}_{\mathrm{PROP}} \to \text{illative contexts}$$

are defined by the following table. (Note that these illative contexts are effectively grammatical conditions on the variables (propositional, individual) that appear in a proposition.)

φ	$[\varphi]^1_r$	$\Gamma^1_r(\varphi)$	$[\varphi]^2_r$	$\Gamma^2_r(\varphi)$
p	rp	$\mathbf{H}(rp)$	rp	$\mathbf{L}(rp)$
$\psi \supset \chi$	$[\psi]^1_r \supset [\chi]^1_r$	$\Gamma^1_r(\psi), \Gamma^1_r(\chi)$	$\mathbf{F}[\psi]^2_r[\chi]^2_r$	$\Gamma^2_r(\psi), \Gamma^2_r(\chi)$

The r in the above can be replaced by I (i.e. omitted). However r in 2.5 is needed and in Proposition B2.15 we used it to derive a relation between the two interpretations.

2.5. Definition (Definition B2.12; interpretation of PRED). (i) $\Lambda^s(\mathbf{\Xi}, \mathbf{L})$ is $\Lambda(\mathbf{\Xi}, \mathbf{L})$ extended by the extra constants A_1, A_2, P, f, g, a associated with the signature s of the manysorted structure of our example. Because we are going to interprete many sorted predicate logic with sorts $\mathbf{A}_1, \mathbf{A}_2$, it is useful to have among the free variables of the λ-calculus infinite sets $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_i = \{x_i, y_i, z_i \ldots\}$. x, y, z denote arbitrary variables.

(ii) Let r be a closed term in $\Lambda^s(\mathbf{\Xi}, \mathbf{L})$. Two maps (for $i = 1, 2$)

$$[-]^i_r : \mathbb{F}_{\mathrm{PRED}} \to \Lambda^s(\mathbf{\Xi}, \mathbf{L})$$

and a map

$$\Gamma : \mathbb{F}_{\mathrm{PRED}} \to \text{illative contexts}$$

are defined by the following tables:

t	$[t]^i_r$	$\Gamma(t)$
$x^{\mathbf{A}_j}$	x_j	$A_j x_j$
$\mathbf{a}$	a	$\emptyset$
$\mathbf{f}s$	$f[s]^i_r$	$\Gamma(s)$
$\mathbf{g}st$	$g[s]^i_r[t]^i_r$	$\Gamma(s), \Gamma(t)$

φ	$[\varphi]^1_r$	$[\varphi]^2_r$	$\Gamma(\varphi)$
$\mathbf{P}t$	$r(P[t]^1_r)$	$r(P[t]^2_r)$	$\Gamma(t)$
$\psi \supset \chi$	$[\psi]^1_r \supset [\chi]^1_r$	$\mathbf{F}[\psi]^2_r[\chi]^2_r$	$\Gamma(\psi), \Gamma(\chi)$
$\forall x^{\mathbf{A}_i}\psi$	$\mathbf{\Xi}A_i(\lambda x_i.[\psi]^1_r)$	$\mathbf{G}A_i(\lambda x_i.[\psi]^2_r)$	$\Gamma(\psi) - \{A_i x_i\}$

(iii)

$$\Gamma^1_{r,s} =< \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1A_1f, \mathbf{F}A_1(\mathbf{F}A_2A_1)g, \mathbf{F}A_1\mathbf{H}(r\circ P), A_1a >$$

,

$$\Gamma^2_{r,s} =< \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1A_1f, \mathbf{F}A_1(\mathbf{F}A_2A_1)g, \mathbf{F}A_1\mathbf{L}(r\circ P), A_1a >$$

and

$$\Gamma^{i,+}_{r,s} = \Gamma^i_{r,s} \cup \{A_2x_2\} \text{ where } x_2 \in \mathcal{V}_2 \text{ is some variable.}$$

The definitions $\Gamma^i_{r,s}$ and $\Gamma^{i,+}_{r,s}$ of course refer to our example of a many-sorted predicate calculus with signature s.

3. Summary of results in preceding and present paper

We had as aim in the preceding paper to prove

(i) $\Delta \vdash_{\text{PROP}} \varphi \Leftrightarrow \forall r[[\Delta]^1_r, \Gamma^1_r(\Delta, \varphi) \vdash_{\mathcal{I}\text{P}} [\varphi]^1_r]$;
(ii) $\Delta \vdash_{\text{PRED}} \varphi \Leftrightarrow \forall r[\Gamma^{1,+}_{r,s}, [\Delta]^1_r, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_r]$;
(iii) $\Delta \vdash_{\text{PROP}} \varphi \Leftrightarrow \forall r \exists M[[\Delta]^2_r, \Gamma^2_r(\Delta, \varphi) \vdash_{\mathcal{I}\text{F}} [\varphi]^2_r M]$;
(iv) $\Delta \vdash_{\text{PRED}} \varphi \Leftrightarrow \forall r \exists M[\Gamma^{2,+}_{r,s}, [\Delta]^2_r, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\text{G}} [\varphi]^2_r M]$.

The proofs of soundness ($\Rightarrow$)were easy. (cf. Propositions B2.11 and B2.14). In Propositions B3.14 and B3.11 we proved completeness for the 2 direct translations $[\ \ ]^1 : \mathrm{PROP} \to \mathcal{I}\mathbf{P}$ and $[\ \ ]^1 : \mathrm{PRED} \to \mathcal{I}\mathbf{\Xi}$, ((i)$\Leftarrow$ and (ii)$\Leftarrow$), by specialising to $r = \mathbf{I}$ i.e.

$$[\Delta]^1_{\mathbf{I}}, \Gamma^1_{\mathbf{I}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{P}} [\varphi]^1_{\mathbf{I}} \ \Rightarrow\ \Delta \vdash_{\mathrm{PROP}} \varphi \quad \text{in B3.14}$$

$$\Gamma^{1,+}_{\mathbf{I},s}, [\Delta]^1_{\mathbf{I}}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_{\mathbf{I}} \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi \quad \text{in B3.11.}$$

In the present paper we prove completeness for the 2 indirect translations $[\ \ ]^2 : \mathrm{PROP} \to \mathcal{I}\mathbf{F}$ and $[\ \ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathbf{G}$, ((iii)$\Leftarrow$ and (iv)$\Leftarrow$), by specialising to $r = \mathbf{K}$. i.e.

$$[\Delta]^2_{\mathbf{K}}, \Gamma^2_{\mathbf{K}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_{\mathbf{K}} M \ \Rightarrow\ \Delta \vdash_{\mathrm{PROP}} \varphi$$

$$\Gamma^{2,+}_{\mathbf{K},s}, [\Delta]^2_{\mathbf{K}}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^2_{\mathbf{K}} M \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

The completeness proof for the direct translation $[\ \ ]^1 : \mathrm{PRED} \to \mathcal{I}\mathbf{\Xi}$ ((ii)$\Leftarrow$) was given in the preceding paper in the following way. We had to show

$$\Gamma' \vdash_{\mathcal{I}\Xi} [\varphi]^1_{\mathbf{I}} \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

where $\Gamma' = \Gamma^{1,+}_{\mathbf{I},s}, [\Delta]^1_{\mathbf{I}}, \Gamma(\Delta, \varphi)$. First we defined in B3.2 a set $\mathcal{O}$ in order to analyze the terms M such that $\Gamma' \vdash_{\mathcal{I}\Xi} M$. We defined $\overline{\mathcal{O}} = \{M \mid \exists N \in \mathcal{O} \mid N =_{\beta\eta} M\}$ and it was clear that $\Gamma' \subseteq \overline{\mathcal{O}}$. We showed in Proposition B3.10 that $\mathcal{O}$ is an invariant (modulo $\beta\eta$-equality) for derivations in $\mathcal{I}\mathbf{\Xi}$:

$$\Gamma \vdash_{\mathcal{I}\Xi} M, \Gamma \subseteq \overline{\mathcal{O}} \Rightarrow M \in \overline{\mathcal{O}}.$$

Using this result we then proved in Proposition B3.11

$$\Gamma' \vdash_{\mathcal{I}\Xi} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

The completeness proof for $[\ \ ]^1 : \mathrm{PROP} \to \mathcal{I}\mathbf{P}$ ((i)$\Leftarrow$) was done in a similar but easier way.

In the present paper we will prove completeness of $[\ \ ]^2 : \mathrm{PROP} \to \mathcal{I}\mathbf{F}$ and $[\ \ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathbf{G}$.

Completenes of $\mathcal{I}\mathbf{F}$ for PROP is relatively easy. We must show

$$[\Delta]^2_{\mathbf{K}}, \Gamma^2_{\mathbf{K}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_{\mathbf{K}} M \ \Rightarrow\ \Delta \vdash_{\mathrm{PROP}} \varphi,$$

i.e. (by Proposition B2.15: $\mathbf{K}[\varphi]^1_r = [\varphi]^2_{\mathbf{K} \circ r}$)

$$[\Delta]^1_{\mathbf{I}}, \Gamma^1_{\mathbf{I}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^1_{\mathbf{I}} \ \Rightarrow\ \Delta \vdash_{\mathrm{PROP}} \varphi.$$

Let us denote the grammar for the completeness proof of $\mathcal{I}\mathbf{P}$ for PROP (defined in B3.12) by $\mathcal{O}_0$ in order to distinguish it from $\mathcal{O}$ above. We shall prove in Proposition 4.3

$$\Gamma \vdash_{\mathcal{I}\mathrm{F}} M,\ \Gamma \subseteq \overline{\mathcal{O}_0} \ \Rightarrow\ \Gamma \vdash_{\mathcal{I}\mathrm{P}} M.$$

Then completenes of $\mathcal{I}\mathbf{F}$ for PROP for the second interpretation follows directly from completenes of $\mathcal{I}\mathbf{P}$ for PROP for the first interpretation.

Completeness of $\mathcal{I}\mathbf{G}$ for PRED is a little bit more complicated. Again by Proposition B2.15 ($\mathbf{K}[\varphi]^1_r = [\varphi]^2_{\mathbf{K}\circ r}$), we see that we must prove

$$\Gamma^{1,+}_{\mathbf{I},s}, [\Delta]^1_{\mathbf{I}}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^1_{\mathbf{I}} \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}} \varphi.$$

We define a grammar, denoted by $\mathcal{O}_1$ to distinguish it from $\mathcal{O}$, and we will show in Proposition 5.11 that $\overline{\mathcal{O}_1}$ is closed under derivations in $\mathcal{I}\mathbf{G}$:

$$\Gamma \vdash_{\mathcal{I}\mathrm{G}} M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M \in \overline{\mathcal{O}_1}.$$

The grammar $\mathcal{O}_1$ is more complicated than $\mathcal{O}$ and we cannot prove completeness directly using this grammar, as in B3.11 for $\mathcal{I}\mathbf{\Xi}$ relative to PRED. But we shall use $\mathcal{O}_1$ in order to show in Lemma 5.19 that in a proof of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$, where $\Gamma \subseteq \overline{\mathcal{O}}$, all cut formulas can be eliminated. From this we get in Proposition 5.23 conservativity of $\mathcal{I}\mathbf{G}$ over $\mathcal{I}\mathbf{\Xi}$

$$\Gamma \subseteq \overline{\mathcal{O}}, M \in \overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M$$

and then completeness of $\mathcal{I}\mathbf{G}$ for PRED is an immediate corollary of this proposition and Proposition B3.11 (completeness of $\mathcal{I}\mathbf{\Xi}$ for PRED).

4. Completeness of $\mathcal{I}\mathbf{F}$ for PROP

We must show for $\varphi \in \mathbb{F}_{\mathrm{PROP}},\ \Delta \subseteq \mathbb{F}_{\mathrm{PROP}}$:

$$\forall r \exists M \in \Lambda [[\Delta]^2_r, \Gamma^2_r(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_r M] \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi.$$

It is sufficient to show

$$\exists M \in \Lambda [[\Delta]^2_{\mathbf{K}}, \Gamma^2_{\mathbf{K}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_{\mathbf{K}} M] \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi.$$

We have by Proposition B2.15 the following relation between the two interpretations:

$$\mathbf{K}[\varphi]^1_r = [\varphi]^2_{\mathbf{K}\circ r}.$$

Hence we must prove

$$[\Delta]^1_{\mathbf{I}}, \Gamma^1_{\mathbf{I}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^1_{\mathbf{I}} \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi.$$

We omit $\mathbf{I}$ in the notation and we prove

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi. \qquad (*)$$

$(*)$ could be proved in exactly the same way as Proposition B3.14 (Completeness for $\mathcal{I}\mathbf{P}$ relative to PROP), changing only $\mathcal{I}\mathbf{P}$ in $\mathcal{I}\mathbf{F}$. The reason for this is the

following. In the proof of Proposition B2.7 we showed that P_e, P_i and P_H follow from F_e, F_i and F_L by the substitutions of $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y. An inspection of the proof of completeness for $\mathcal{IP}$ relative to PROP shows that every X and Y occurring in the proof in cases P_e, P_i and P_H have the form $\mathbf{K}X_1$ and $\mathbf{K}Y_1$ and therefore also a substitution in the opposite direction is possible.

We will give another proof and show that (*) is a *consequence* of Proposition B3.14 by proving in Proposition 4.3 that we have

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{IF}} [\varphi]^1 \;\Rightarrow\; [\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{IP}} [\varphi]^1.$$

4.1. Definition. If $\mathcal{R}$ is any grammatical class in this paper then

$$\overline{\mathcal{R}} = \{X \mid (\exists Y) Y \in \mathcal{R} \wedge X =_{\beta\eta} Y\}.$$

4.2. Definition (Grammar for derivable formulas for $\mathcal{IF}$). We copy Definition B3.12, adding the index 0.

$$\begin{array}{rcl} \mathcal{P}_0 & = & \mathcal{V}_0 \mid \mathcal{P}_0 \supset \mathcal{P}_0 \equiv \mathbf{\Xi}(\mathbf{K}\mathcal{P}_0)(\mathbf{K}\mathcal{P}_0); \\ \mathcal{G}_0 & = & \mathbf{H}\mathcal{P}_0 \mid \mathcal{P}_0 \supset \mathcal{G}_0 \equiv \mathbf{\Xi}(\mathbf{K}\mathcal{P}_0)(\mathbf{K}\mathcal{G}_0); \\ \mathcal{O}_0 & = & \mathcal{G}_0 \mid \mathcal{P}_0. \end{array}$$

Note that $[\Delta]^1, \Gamma^1(\Delta, \varphi) \subseteq \overline{\mathcal{O}_0}$.

4.3. Proposition. *$\mathcal{IF}$ is conservative over $\mathcal{IP}$ in the following sense*

$$\Gamma \vdash_{\mathcal{IF}} M \;\&\; \Gamma \subseteq \overline{\mathcal{O}_0} \;\Rightarrow\; \Gamma \vdash_{\mathcal{IP}} M \;\&\; M \in \overline{\mathcal{O}_0}. \qquad (**)$$

Proof. We rewrite the rules for $\mathcal{IF}$ in the following way:

$$\begin{array}{lllll} \mathsf{F}_\mathrm{e} & \Gamma \vdash \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash XV & \Rightarrow & \Gamma \vdash (Y \circ Z)V; \\ \mathsf{F}_\mathrm{i} & \Gamma, Xx \vdash (Y \circ Z)x,\ \Gamma \vdash \mathbf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y, Z) & \Rightarrow & \Gamma \vdash \mathbf{\Xi}X(Y \circ Z); \\ \mathsf{F}_\mathsf{L} & \Gamma, Xx \vdash \mathbf{L}Y,\ \Gamma \vdash \mathbf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y) & \Rightarrow & \Gamma \vdash \mathbf{L}(\mathbf{F}XY). \end{array}$$

The proof of (**) is by induction on the proof of $\Gamma \vdash_{\mathcal{IF}} M$.

Case F_e. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} (Y \circ Z)V$ as consequence of $\Gamma \vdash_{\mathcal{IF}} \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash_{\mathcal{IF}} XV$.

By the induction hypothesis one has

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash_{\mathcal{IP}} XV,$$

$$\mathbf{\Xi}X(Y \circ Z) = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O),\ XV = p,$$

where $p \in \mathcal{P}_0$ and $O \in \mathcal{O}_0$. By rule $\mathsf{P_e}$ we have $\Gamma \vdash_{\mathcal{IP}} O$, where $O = (Y \circ Z)V = M$.

Case $\mathsf{F_i}$. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} \mathbf{\Xi}X(Y \circ Z)$ as consequence of
$\Gamma \vdash_{\mathcal{IF}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{IF}} (Y \circ Z)x$ with $x \notin \mathrm{FV}(\Gamma, X, Y, Z)$.

By the IH we have

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{L}X, \quad X = \mathbf{K}p, \quad \Gamma, p \vdash_{\mathcal{IP}} (Y \circ Z)x$$

and $(Y \circ Z)x = O \in \mathcal{O}_0$. Now $x \notin \mathrm{FV}(\Gamma, p)$, so $x \notin \mathrm{FV}(O)$ by Proposition B3.10, hence $Y \circ Z = \mathbf{K}O$. Therefore by rule $\mathsf{P_i}$ we get

$$\Gamma \vdash_{\mathcal{IP}} p \supset O$$

and $p \supset O = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O) = \mathbf{\Xi}X(Y \circ Z) = M$.

Case $\mathsf{F_L}$. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} \mathbf{L}(\mathbf{F}XY)$ as consequence of
$\Gamma \vdash_{\mathcal{IF}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{IF}} \mathbf{L}(Y)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{L}X, \quad X = \mathbf{K}p, \quad \Gamma, p \vdash_{\mathcal{IP}} \mathbf{L}(Y), \quad Y = \mathbf{K}p_1.$$

Hence by rule $\mathsf{P_H}$ $\Gamma \vdash_{\mathcal{IP}} \mathbf{H}(p \supset p_1)$. Now $\mathbf{H}(p \supset p_1) = \mathbf{L}(\mathbf{K}(p \supset p_1)) = \mathbf{L}(\mathbf{F}(\mathbf{K}p)(\mathbf{K}p_1)) = \mathbf{L}(\mathbf{F}XY)$. So $\Gamma \vdash_{\mathcal{IP}} M$. □

5. Completeness of $\mathcal{IG}$ for PRED

We will show for $\varphi \in \mathbb{F}_{\mathrm{PRED}}$, $\Delta \subseteq \mathbb{F}_{\mathrm{PRED}}$:

$$\forall r[\ \Gamma^{2,+}_{r,s}, [\Delta]^2_r, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^2_r M \text{ some } M\ ] \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

It is sufficient to show

$$\Gamma^{2,+}_{\mathbf{K},s}, [\Delta]^2_{\mathbf{K}}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^2_{\mathbf{K}} M \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi,$$

i.e. (by Proposition B2.15: $\mathbf{K}[\varphi]^1_r = [\varphi]^2_{\mathbf{K} \circ r}$)

$$\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

Here $\Gamma^{1,+}_s, [\Delta]^1$ and $[\varphi]^1$ stand for $\Gamma^{1,+}_{\mathbf{I},s}, [\Delta]^1_{\mathbf{I}}$ and $[\varphi]^1_{\mathbf{I}}$.

The proof is again in several steps. First we define in 5.2 a *grammar* in order to analyze the terms N such that $\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N$. Using this grammar we show in Lemma 5.19 that in a proof of $\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N$ all cuts can be eliminated . From this we get in Proposition 5.23

$$\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N \ \Rightarrow\ \Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I\Xi}} N$$

and then completeness of $\mathcal{I}\mathbf{G}$ for PRED is an immediate corollary of this proposition and Proposition B3.11. Instead of $\vdash_{\mathcal{I}\mathrm{G}}$ we shall mostly write $\vdash$.

Now first we repeat Definition B3.2, the grammar for derivable statements in $\mathcal{I}\mathbf{\Xi}$ for the manysorted structure of our example, in order to contrast it with the grammar for derivable statements in $\mathcal{I}\mathbf{G}$ of Definition 5.2 and also because we need it in the sequel. Instead of $x_i, \mathcal{V}_i$ and $\mathcal{T}_i$ we shall write $x^{A_i}, \mathcal{V}^{A_i}$ and $\mathcal{T}^{A_i}$.

5.1. Definition (Grammar for derivable statements in $\mathcal{I}\mathbf{\Xi}$).

$$\begin{aligned}
&\mathcal{T} = \mathcal{T}^{A_1} \mid \mathcal{T}^{A_2};\\
&\mathcal{T}^{A_1} = \mathcal{V}^{A_1} \mid a \mid f\mathcal{T}^{A_1} \mid g\mathcal{T}^{A_1}\mathcal{T}^{A_2};\\
&\mathcal{T}^{A_2} = \mathcal{V}^{A_2};\\
\\
&\mathcal{P} = P\mathcal{T}^{A_1} \mid \mathbf{\Xi}A_i(\lambda x^{A_i}.\mathcal{P}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P});\\
&\mathcal{G} = \mathbf{L}A_i \mid A_i\mathcal{T}^{A_i} \mid \mathbf{\Xi}A_i(\lambda x^{A_i}.\mathcal{G}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}) \mid \mathbf{L}(\mathbf{K}\mathcal{P});\\
&\mathcal{O} = \mathcal{G} \mid \mathcal{P}.
\end{aligned}$$

Now we will define the grammar $\mathcal{O}_1$ for derivable statements in $\mathcal{I}\mathbf{G}$. $\mathcal{O}_1$ is more complicated then the grammar $\mathcal{O}$ for $\mathcal{I}\mathbf{\Xi}$ for several reasons. Write Γ' for $\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi)$. If $\Gamma' \vdash \mathbf{L}X$ in system $\mathcal{I}\mathbf{\Xi}$ then $X = A_i$ or $X = \mathbf{K}p$ with $p \in \mathcal{P}$. But if $\Gamma' \vdash \mathbf{L}X$ in system $\mathcal{I}\mathbf{G}$ then we can have for example $X = \mathbf{F}A_1A_2$ or even $X = \mathbf{F}(\mathbf{K}p)A_2$. Therefore we define an extension $\mathcal{L}_1$ of the set of sorts $\{A_1, A_2\}$ containing these X's. We need for each $l \in \mathcal{L}_1$ a set of variables $\mathcal{V}_1^l$, similar to the sets $\mathcal{V}^{A_1}$ and $\mathcal{V}^{A_2}$. If we took a different set of variables for each $l \in \mathcal{L}_1$, then the substitution lemma 5.5, that is needed in the proof of Proposition 5.11, would not hold. Therefore we define for each $l \in \mathcal{L}_1$ the skeleton $\mathrm{sk}(l)$, such that $sk(\mathbf{K}p)$ is the same for all $p \in \mathcal{P}_1$ and we define $\mathcal{V}_1^{l_1} = \mathcal{V}_1^{l_2}$ iff $sk(l_1) = sk(l_2)$.

The sets $\mathcal{T}_1^l$ will contain $\lambda-$abstractions: if $l = \mathbf{F}l_1l_2$ and $t^{l_2} \in \mathcal{T}_1^{l_2}$ then $\lambda x^{l_1}.t^{l_2} \in \mathcal{T}_1^l$. This may seem a little bit strange, but in fact it is very natural. For example we have $\vdash \mathbf{F}A_1A_1(\lambda x^{A_1}.x^{A_1})$ by rule F_i. Hence $\lambda x^{A_1}.x^{A_1}$ should be in $\mathcal{T}_1^{\mathbf{F}A_1A_1}$. Moreover in the proof of Lemma 5.9 we really need that $\lambda x^{l_1}.t^{l_2} \in \mathcal{T}_1^{\mathbf{F}l_1l_2}$.

Finally $\mathcal{P}_1, \mathcal{G}_1$ and $\mathcal{O}_1$ are very similar to $\mathcal{P}, \mathcal{G}$ and $\mathcal{O}$, except that besides $\mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O) \in \mathcal{O}_1$ we now also have $\mathbf{\Xi}(\mathbf{K}p)(\lambda x^{\mathbf{K}p}.O) \in \mathcal{O}_1$, where $x^{\mathbf{K}p}$ may occur free in O.

5.2. Definition (Grammar for derivable statements in $\mathcal{I}\mathbf{G}$). (i) $\Lambda^s(\mathbf{\Xi}, \mathbf{L})$ is, as in Definition B2.12, $\Lambda(\mathbf{\Xi}, \mathbf{L})$ extended by the extra constants A_1, A_2, P, f, g, a associated with the signature s of the manysorted structure of our example.

(ii) Now we will define simultaneously sets $\mathcal{Q}_1$ and $\mathcal{L}_1$ of sorts (extensions of the set $\{A_1, A_2\}$), for each $l \in \mathcal{L}_1$ the skeleton $\mathrm{sk}(l)$, a set of variables $\mathcal{V}_1^l$ and a set of terms $\mathcal{T}_1^l$, sets $\mathcal{P}_1, \mathcal{G}_1$ and $\mathcal{O}_1$.

Definition of $\mathcal{Q}_1$ *and* $\mathcal{L}_1$.

$$\begin{aligned}&\mathcal{Q}_1 = A_1 \mid A_2 \mid \mathbf{G}l(\lambda x^l.q) \quad (l{\in}\mathcal{L}_1, x^l{\in}\mathcal{V}_1^l, q{\in}\mathcal{Q}_1);\\ &\mathcal{L}_1 = \mathcal{Q}_1 \cup \mathbf{K}\mathcal{P}_1.\end{aligned}$$

Definition of $sk(l)$.

$$\begin{aligned}&sk(A_i) = A_i;\\ &sk(\mathbf{K}p) = \mathbf{K}(Px_1^{A_1}) \quad \text{(i.e. } sk(\mathbf{K}p) \text{ is independent of } p);\\ &sk(\mathbf{G}l(\lambda x^l.q)) = \mathbf{G}(sk(l))(\lambda x^l.sk(q)).\end{aligned}$$

Definition of $\mathcal{V}_1^l$.

We assume that for each skeleton $sk(l)$ there is a different set of variables

$$\mathcal{V}_1^{sk(l)} = \{x^{sk(l)}, x_1^{sk(l)}, x_2^{sk(l)}, \dots y^{sk(l)}, y_1^{sk(l)}, y_2^{sk(l)}, \dots\}$$

and for each $l \in \mathcal{L}_1$ we define

$$\mathcal{V}_1^l = \mathcal{V}_1^{sk(l)}, \quad x^l = x^{sk(l)}, x_i^l = x_i^{sk(l)}, \quad y^l = y^{sk(l)}, y_i^l = y_i^{sk(l)}.$$

Definition of $\mathcal{T}_1^l$.

$$\begin{aligned}&\mathcal{T}_1^l = a \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathbf{F}l_1l}\mathcal{T}_1^{l_1}, && \text{if } l \equiv A_1;\\ &\mathcal{T}_1^l = \quad \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathbf{F}l_1l}\mathcal{T}_1^{l_1}, && \text{if } l \equiv A_2;\\ &\mathcal{T}_1^l = f \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathbf{F}l_1l}\mathcal{T}_1^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_1^{A_1}, && \text{if } l \equiv \mathbf{F}A_1A_1;\\ &\mathcal{T}_1^l = g \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathbf{F}l_1l}\mathcal{T}_1^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_1^{\mathbf{F}A_2A_1}, && \text{if } l \equiv \mathbf{F}A_1(\mathbf{F}A_2A_1);\\ &\mathcal{T}_1^l = \quad \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathbf{G}l_1(\lambda x^{l_1}.l)}\mathcal{T}_1^{l_1} \mid \lambda x^{l_2}.\mathcal{T}_1^{l_3}, && \text{if } l \equiv \mathbf{G}l_2(\lambda x^{l_2}.l_3), l{\notin}\{\mathbf{F}A_1A_1, \mathbf{F}A_1(\mathbf{F}A_2A_1)\};\\ &\mathcal{T}_1^l = \Lambda^s(\mathbf{\Xi}, \mathbf{L}), && \text{if } l \equiv \mathbf{K}p.\end{aligned}$$

Definition of $\mathcal{P}_1, \mathcal{G}_1$ *and* $\mathcal{O}_1$.

$$\begin{aligned}&\mathcal{P}_1 = P\mathcal{T}_1^{A_1} \mid \mathbf{\Xi}l(\lambda x^l.\mathcal{P}_1);\\ &\mathcal{G}_1 = \mathbf{L}\mathcal{L}_1 \mid A_i\mathcal{T}_1^{A_i} \mid \mathbf{\Xi}l(\lambda x^l.\mathcal{G}_1);\\ &\mathcal{O}_1 = \mathcal{P}_1 \mid \mathcal{G}_1.\end{aligned}$$

(iii) *Notation.*

$$\begin{aligned}&q, q_1, q_2, q', \dots \text{ denote elements of } \mathcal{Q}_1;\\ &l, l_1, l_2, l', \dots \text{ denote elements of } \mathcal{L}_1;\\ &p, p_1, p_2, p', \dots \text{ denote elements of } \mathcal{P}_1.\end{aligned}$$

5.3. Remarks. (i) $\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \subseteq \overline{\mathcal{O}} \subseteq \overline{\mathcal{O}_1}$.

(ii) If $\mathrm{FV}(q)=\emptyset$ then $\mathbf{G}l(\lambda x^l.q)=_\beta \mathbf{F}lq$. In that case we will denote $\mathbf{G}l(\lambda x^l.q)$ usually by $\mathbf{F}lq$. In fact we did that already in the above definition.

(iii) The grammar in the definition is not context-free and this causes some inaccuracies in the notation. In the definition of $\mathcal{P}_1$ (and similarly at some other places) instead of $\mathbf{\Xi}l(\lambda x^l.\mathcal{P}_1)$ we should write $\mathbf{\Xi}\mathcal{L}_1(\lambda x^{\mathcal{L}_1}.\mathcal{P}_1)$, but then we do not know that these two $\mathcal{L}_1$'s represent the *same* $l{\in}\mathcal{L}_1$. (We may not have $\mathbf{\Xi}l_1(\lambda x^{l_2}.\mathcal{P}_1)$.)

(iv) In the definition of $\mathcal{T}_1^l$ instead of the clause $\mathcal{T}_1^{\mathbf{G}l_1(\lambda x^{l_1}.l)}\mathcal{T}_1^{l_1}$ one might expect, in view of rule G_e a clause $t_1^{\mathbf{G}l_1(\lambda x^{l_1}.l')}t_2^{l_1}$ where $l=l'[x^{l_1}:=t_2^{l_1}]$, but that does not make a difference because we have for $l{\in}\mathcal{L}_1$

$$\mathcal{T}_1^{l[x^{l_1}:=t^{l_1}]}=\mathcal{T}_1^l,$$

as follows easily from

$$\mathcal{V}_1^{l[x^{l_1}:=t^{l_1}]}=\mathcal{V}_1^l$$

in Lemma 5.5.

5.4. Lemma. $l_1,l_2{\in}\mathcal{L}_1 \;\Rightarrow\; \mathbf{G}l_1(\lambda x^{l_1}.l_2){\in}\mathcal{L}_1$.

Proof. It is clear if $l_2{\in}\mathcal{Q}_1$. Now let $l_2=\mathbf{K}p$. Then
$\mathbf{G}l_1(\lambda x^{l_1}.l_2)=\lambda z.\mathbf{\Xi}l_1(\mathbf{S}(\lambda x^{l_1}.\mathbf{K}p)z)=\lambda z.\mathbf{\Xi}l_1(\lambda x^{l_1}.p)=\mathbf{K}(\mathbf{\Xi}l_1(\lambda x^{l_1}.p)){\in}\mathbf{K}\mathcal{P}_1$. □

5.5. Lemma. *Let* $\mathcal{W}=\mathcal{Q}_1,\mathcal{L}_1,\mathcal{T}_1^l,\mathcal{P}_1,\mathcal{G}_1,$ *or* $\mathcal{O}_1$ *and let* $w{\in}\mathcal{W}, x^{l_1}{\in}\mathcal{V}_1^{l_1}, t^{l_1}{\in}\mathcal{T}_1^{l_1}$, *then*

$$w[x^{l_1}:=t^{l_1}]{\in}\mathcal{W},$$

moreover if $\mathcal{W}=\mathcal{L}_1$ *then*

$$sk(w[x^{l_1}:=t^{l_1}])=sk(w).$$

and hence

$$\mathcal{V}_1^{w[x^{l_1}:=t^{l_1}]}=\mathcal{V}_1^w.$$

Proof. By induction on the structure of w. Instead of x^{l_1} and t^{l_1} we shall write x and t.

Case $w{\in}\mathcal{Q}_1$.

$w=A_i$. This is trivally ok.

$w=\mathbf{G}l_2(\lambda y^{l_2}.q)$. Then $w[x:=t]=\mathbf{G}(l_2[x:=t])(\lambda y^{l_2}.q[x:=t])=$
$=\mathbf{G}(l_2[x:=t])(\lambda y^{l_2[x:=t]}.q[x:=t])$
(by the second part of the IH for l_2) and this is in $\mathcal{Q}_1$ by the first part of the IH for l_2 and q. The second part of the IH for l_2 and q yields $sk(w[x:=t])=sk(w)$.

Case $w \in \mathcal{L}_1$.
This follows directly from the cases $w \in \mathcal{Q}_1, w \in \mathcal{P}_1$.

Case $w = t^l \in \mathcal{T}_1^l$.

$l = A_1$. Now $w = x_1^{A_1}, a$ or $t_1^{\mathbf{G}l(\lambda x_1^l.A_1)} t_2^l$. All these cases are easy.
$l = A_2$. Easy.
$l = \mathbf{G}l_2(\lambda y^{l_2}.q)$. Now $w = x_1^l, t_1^{\mathbf{G}l_3(\lambda x_1^{l_3}.l)} t_2^{l_3}, \lambda y^{l_2}.t_1^q, f$ or g. Easy again.
$l = \mathbf{K}p$. Then $w[x := t] \in \Lambda = \mathcal{T}_1^{\mathbf{K}p}$.

Case $w \in \mathcal{P}_1$.

$w = Pt_1^{A_1}$. Then $w[x := t] = P(t_1^{A_1}[x := t])$.
$w = \mathbf{\Xi}l_2(\lambda y^{l_2}.p)$. Then $w[x := t] = \mathbf{\Xi}(l_2[x := t])(\lambda y^{l_2}.p[x := t]) =$
$= \mathbf{\Xi}(l_2[x := t])(\lambda y^{l_2[x:=t]}.p[x := t])$.

Case $w \in \mathcal{G}_1$. Similarly. □

5.6. Lemma. *Let* $cX_1 \dots X_n =_{\beta\eta} M$ *for some constant* c *and some* $M \in \mathcal{O}_1$. *Then* $n \in \{1, 2\}$ *and* $M \equiv cY_1 \dots Y_n$ *with* $Y_i =_{\beta\eta} X_i$.

Proof. By Church-Rosser for $\beta\eta$-reduction. □

5.7. Lemma. *Let* $q \in \mathcal{Q}_1$ $p \in \mathcal{P}_1$, $M \in \Lambda^s(\mathbf{\Xi}, \mathbf{L})$ *and let* σ *substitute variables for variables. Then* $qM \neq_{\beta\eta} p^\sigma$.

Proof. By an easy induction on q, using Church-Rosser. □

In the definition of $\mathcal{G}_1$ one of the clauses is $A_i \mathcal{T}_1^{A_i}$ where one might expect the more general clause $q\mathcal{T}_1^q$. But $A_i \mathcal{T}_1^{A_i}$ is sufficient as follows from the next lemma.

5.8. Lemma.

$$qt^q \in \overline{\mathcal{G}_1} \text{ for each } q \in \mathcal{Q}_1.$$

Proof. By induction on the structure of q. It is clear if $q = A_1$ or A_2. So let $q = \mathbf{G}l_1(\lambda x^{l_1}.q_1)$. Then $qt^q = \mathbf{G}l_1(\lambda x^{l_1}.q_1)t^q = \mathbf{\Xi}l_1(\mathbf{S}(\lambda x^{l_1}.q_1)t^q) = \mathbf{\Xi}l_1(\lambda x^{l_1}.q_1(t^q x^{l_1}))$. Now $t^q \in \mathcal{T}_1^q = \mathcal{T}_1^{\mathbf{G}l_1(\lambda x^{l_1}.q_1)}$ and $x^{l_1} \in \mathcal{T}_1^{l_1}$ hence $t^q x^{l_1} \in \mathcal{T}_1^{q_1}$. So by the induction hypothesis one has $q_1(t^q x^{l_1}) \in \overline{\mathcal{G}_1}$. Hence $\mathbf{\Xi}l_1(\lambda x^{l_1}.q_1(t^q x^{l_1})) \in \overline{\mathcal{G}_1}$. □

5.9. Lemma.

$$lV \in \overline{\mathcal{O}_1} \;\Rightarrow\; V \in \overline{\mathcal{T}_1^l}.$$

Proof. By induction on the structure of l. It is clear if $l = A_1, A_2$ or $\mathbf{K}p$. So let $l = \mathbf{G}l_1(\lambda x^{l_1}.q)$. Then $lV = \mathbf{\Xi}l_1(\lambda x^{l_1}.q(Vx^{l_1}))$. So $q(Vx^{l_1}) \in \overline{\mathcal{O}_1}$ and hence by the induction hypothesis we have $Vx^{l_1} = t^q \in \mathcal{T}_1^q$. So $V = \lambda x^{l_1}.t^q \in \mathcal{T}_1^l$. □

5.10. Remark. Clearly $(\mathsf{K}p)t^{\mathsf{K}p} = p \in \mathcal{P}_1$, hence combining Lemma's 5.8 and 5.9 we get

$$lV \in \overline{\mathcal{O}_1} \;\Leftrightarrow\; V \in \overline{\mathcal{T}_1^l}.$$

Now we will prove $\Gamma \vdash M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M \in \overline{\mathcal{O}_1}$. The proof is different from the proof of Proposition B3.10, because we do not have

$$\Gamma \vdash M,\ \Gamma \subseteq \overline{\mathcal{O}_1}\ \&\ u \notin_{\beta\eta} \mathrm{FV}(\Gamma) \;\Rightarrow\; u \notin_{\beta\eta} \mathrm{FV}(M).$$

This is shown by the following example

$$\mathsf{F}(\mathsf{K}p)A_1 x,\ p \vdash A_1(xu).$$

In fact this is the reason that in the grammar $\mathcal{O}_1$ we have elements $\mathsf{\Xi}(\mathsf{K}p)(\lambda x^{\mathsf{K}p}.O)$ instead of the simpler elements $\mathsf{\Xi}(\mathsf{K}p)(\mathsf{K}O)$.

5.11. Proposition. *In $\mathcal{I}\mathsf{G}$ we have*

$$\Gamma \vdash M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M \in \overline{\mathcal{O}_1}.$$

Proof. By induction on the derivation of $\Gamma \vdash M$. We only consider the three specific rules $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$ and G_{L}. (The two general rules are easy and $\mathsf{G}_{\mathrm{e}}, \mathsf{G}_{\mathrm{i}}$ are equivalent to $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$.)

Case Ξ_{e}. $\Gamma \vdash M$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \mathsf{\Xi}XY,\ \Gamma \vdash XV$.

By the IH one has $\mathsf{\Xi}XY \in \overline{\mathcal{O}_1}, XV \in \overline{\mathcal{O}_1}$. From $\mathsf{\Xi}XY \in \overline{\mathcal{O}_1}$ it follows that $X = l, Y = \lambda x^l.O$ where $O \in \mathcal{O}_1$. Now $XV = lV \in \overline{\mathcal{O}_1}$, hence $V = t^l \in \mathcal{T}_1^l$ by Lemma 5.9. $M = (\lambda x^l.O)t^l = O[x^l := t^l] \in \mathcal{O}_1$ by Lemma 5.5.

Case Ξ_{i}. $\Gamma \vdash M$ is $\Gamma \vdash \mathsf{\Xi}XY$ as direct consequence of
$\Gamma \vdash \mathsf{L}X,\ \Gamma, Xx \vdash Yx$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has $\mathsf{L}X \in \overline{\mathcal{O}_1}$, hence $X = l \in \mathcal{L}_1$. Now x is any variable, so we may assume that $x = x^l \in \mathcal{V}_1^l$. Then $lx \in \overline{\mathcal{O}_1}$ by Lemma 5.8, hence by the IH we have $Yx \in \overline{\mathcal{O}_1}$. Let $Yx = O \in \overline{\mathcal{O}_1}$. Then $M = \mathsf{\Xi}l(\lambda x^l.O) \in \overline{\mathcal{O}_1}$.

Case G_{L}. $\Gamma \vdash M$ is $\Gamma \vdash \mathsf{L}(\mathsf{G}XY)$ as direct consequence of
$\Gamma \vdash \mathsf{L}X,\ \Gamma, Xx \vdash \mathsf{L}(Yx)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has $\mathsf{L}X \in \overline{\mathcal{O}_1}$, hence $X = l \in \mathcal{L}_1$. Again we may assume that $x = x^l \in \mathcal{V}_1^l$. Then $lx \in \overline{\mathcal{O}_1}$, hence by the IH we have $\mathsf{L}(Yx) \in \overline{\mathcal{O}_1}$. So $Yx = l_1 \in \mathcal{L}_1$. $\mathsf{G}XY = \mathsf{G}l(\lambda x^l.l_1)) \in \overline{\mathcal{L}_1}$ by Lemma 5.4. Hence $\mathsf{L}(\mathsf{G}XY) \in \overline{\mathcal{G}_1} \subseteq \overline{\mathcal{O}_1}$. □

The grammar $\mathcal{O}_1$ is more complicated than $\mathcal{O}$ and we cannot prove completeness directly using this grammar, as in in B3.11 for $\mathcal{I}\mathsf{\Xi}$ relative to PRED. We will show in 5.12-5.21 that in each proof $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$, where $\Gamma \subseteq \overline{\mathcal{O}_1}$, all cut formulas

can be eliminated. From this we get in Proposition 5.23 conservativity of $\mathcal{I}\mathbf{G}$ over $\mathcal{I}\Xi$

$$\Gamma \subseteq \overline{\mathcal{O}}, M{\in}\overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M$$

and then completeness of $\mathcal{I}\mathbf{G}$ for PRED is an immediate corollary of this proposition and Proposition B3.11.

Cut elimination

First we define the Ξ–length $L_\Xi(O)$ for $O{\in}\overline{\mathcal{O}_1}$ (and $L_\Xi(l)$ for $l{\in}\mathcal{L}_1$) and state some lemma's, without the (trivial) proofs.

5.12. Definition (Ξ–length L_Ξ).

$$L_\Xi(Pt^{A_1}) = L_\Xi(\mathbf{L}l) = L_\Xi(A_i t^{A_i}) = 0,\;\; L_\Xi(\Xi l(\lambda x^l.O)) = 1 + L_\Xi(l) + L_\Xi(O),$$

$$L_\Xi(A_i) = 0,\;\; L_\Xi(\mathbf{G}l(\lambda x^l.q)) = 1 + L_\Xi(l) + L_\Xi(q),\;\; L_\Xi(\mathbf{K}p) = L_\Xi(p).$$

5.13. Lemma. (i) $X, Y{\in}\mathcal{L}_1 \cup \mathcal{O}_1,\; X \twoheadrightarrow_{\beta\eta} Y \;\Rightarrow\; L_\Xi(X) = L_\Xi(Y)$.
(ii) $X, Y{\in}\mathcal{L}_1 \cup \mathcal{O}_1,\; X =_{\beta\eta} Y \;\Rightarrow\; L_\Xi(X) = L_\Xi(Y)$.

5.14. Definition. Let $M{\in}\overline{\mathcal{O}_1},\; M =_{\beta\eta} O$. Then $L_\Xi(M) = L_\Xi(O)$.

5.15. Lemma. $L_\Xi(lt^l) = L_\Xi(l)$.

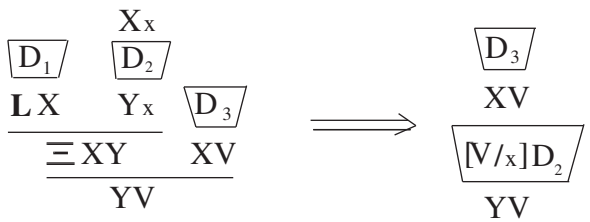
5.16. Lemma. $X{\in}\mathcal{L}_1 \cup \overline{\mathcal{O}_1},\; t^l{\in}\mathcal{T}_1^l \;\Rightarrow\; L_\Xi(X[x^l := t^l]) = L_\Xi(X)$.

5.17. Lemma.

$$\Xi XY{\in}\overline{\mathcal{O}_1},\; XV{\in}\overline{\mathcal{O}_1} \;\Rightarrow\; L_\Xi(XV) < L_\Xi(\Xi XY)\;\&\; L_\Xi(YV) < L_\Xi(\Xi XY).$$

Proof. $\Xi XY = \Xi l(\lambda x^l.O)$ and $V{\in}\overline{\mathcal{T}_1^l}$. Now use Lemma's 5.15 and 5.16.

5.18. Definition. A cut elimination is a proof reduction of the following form

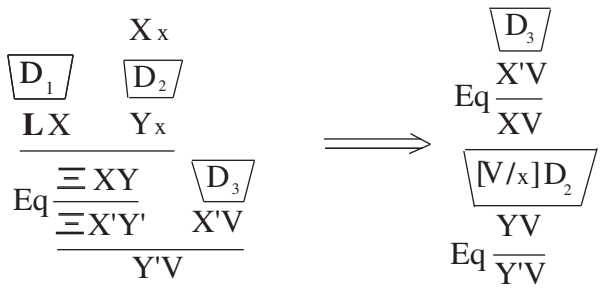


ΞXY is called the cut formula.

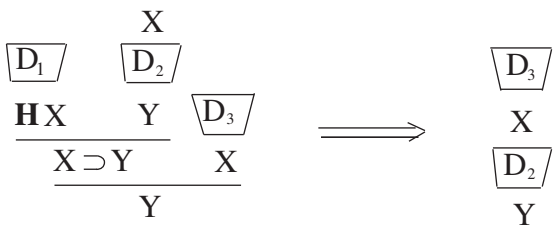
5.19. Lemma. *In a proof of* $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ *(or* $\Gamma \vdash_{\mathcal{I}\Xi} M$ *), where* $\Gamma \subseteq \overline{\mathcal{O}_1}$, *all cut formulas can be eliminated by the above proof reduction.*

PROOF. By induction on the maximal $\Xi-$length of cut formulas in the proof of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$. Consider a cut in the proof with a cut formula ΞXY of maximal $\Xi-$length. Eliminate this cut. Now new cut formulas may have been introduced at XV and YV, however by Lemma 5.17 these have shorter $\Xi-$length. In the same way all other cut formulas of maximal $\Xi-$length can be eliminated and so by the induction hypothesis all cut formulas can be eliminated. □

5.20. REMARK. The above can easily be generalised to allow $\beta\eta-$equality steps (Eq-steps) as follows.



5.21. REMARK. Similarly we can define cut elimination for $\mathcal{I}\mathbf{P}$ as a proof reduction of the following form.



This is in fact a special case of Definition 5.18. (Substitute $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y in Definition 5.18.) Also in a proof of $\Gamma \vdash_{\mathcal{I}\mathrm{P}} M$ where $\Gamma \subseteq \overline{\mathcal{O}_1}$ all cut formulas can be eliminated.

Now we need only one more lemma to derive conservativity of $\mathcal{I}\mathbf{G}$ over $\mathcal{I}\Xi$ in Proposition 5.23.

5.22. LEMMA. *Let $\Gamma \subseteq \overline{\mathcal{O}}$ and let $\Gamma \vdash_{\mathcal{I}\mathrm{G}} N$ be a cut free proof. If the last applied rule in the proof is not the $\Xi_{\mathrm{i}}-$rule, nor the $\mathsf{G}_{\mathsf{L}}-$rule (modulo the $\beta\eta-$rule), then $N \in \overline{\mathcal{O}}$.*

PROOF. The proof is by induction on $\Gamma \vdash N$. The two general cases are easy, so we only consider

Case Ξ_{e}. $\Gamma \vdash N$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \Xi XY$, $\Gamma \vdash XV$.

By the induction hypothesis for $\Gamma \vdash \mathbf{\Xi}XY$ we have $\mathbf{\Xi}XY \in \overline{\mathcal{O}}$. We distinguish two subcases

Subcase a). $\mathbf{\Xi}XY = \mathbf{\Xi}A_i(\lambda x^{A_i}.O)$ with $O \in \overline{\mathcal{O}}$. The induction hypothesis for $\Gamma \vdash A_i V$ yields $A_i V \in \overline{\mathcal{O}}$ and hence $V \in \overline{\mathcal{T}^{A_i}}$. So $N = YV = O[x^{A_i} := V] \in \overline{\mathcal{O}}$.
Subcase b). $\mathbf{\Xi}XY = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O)$ with $O \in \overline{\mathcal{O}}$. Now $YV = O \in \overline{\mathcal{O}}$. □

5.23. Proposition. *$\mathcal{I}\mathbf{G}$ is conservative over $\mathcal{I}\mathbf{\Xi}$ in the following sense*

$$\Gamma \subseteq \overline{\mathcal{O}}, M \in \overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M.$$

Proof. By induction on the derivation of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$. We may assume that the proof is cut free. Again we only consider the three specific rules $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$ and G_{L}.

Case Ξ_{e}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} YV$ as direct consequence of
$\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{\Xi}XY$, $\Gamma \vdash_{\mathcal{I}\mathrm{G}} XV$.

$\mathbf{\Xi}XY \in \overline{\mathcal{O}}$ by Lemma 5.22, so by the IH we have $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{\Xi}XY$. Now $X = A_i$ or $X = \mathbf{K}p_1$, so also $XV \in \overline{\mathcal{O}}$ (use Lemma 5.22 for the case: $X = A_i$). Again by the IH we get $\Gamma \vdash_{\mathcal{I}\Xi} XV$. So $\Gamma \vdash_{\mathcal{I}\Xi} YV$ by rule Ξ_{e}.

Case Ξ_{i}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{\Xi}XY$ as direct consequence of
$\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{I}\mathrm{G}} Yx$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

$\mathbf{\Xi}XY \in \overline{\mathcal{O}}$ so $X = A_i$ or $X = \mathbf{K}p_1$ and $Yx = O \in \overline{\mathcal{O}}$. So $\mathbf{L}X \in \overline{\mathcal{O}}$, hence we have by the IH $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}X$. Now $\Gamma, Xx \subseteq \overline{\mathcal{O}}, Yx \in \overline{\mathcal{O}}$, so $\Gamma, Xx \vdash_{\mathcal{I}\Xi} Yx$ by the IH. Hence $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{\Xi}XY$.

Case G_{L}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}(\mathbf{G}XY)$ as direct consequence of
$\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}(Yx)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

$\mathbf{L}(\mathbf{G}XY) \in \overline{\mathcal{O}}$ so $\mathbf{G}XY = \mathbf{K}p$, where $p = \mathbf{\Xi}A_i(\lambda x^{A_i}.p_2)$ or $p = \mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2)$ with $p_1, p_2 \in \mathcal{P}$. Since $\mathbf{L}X \in \overline{\mathcal{O}}$, we have by the IH $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}X$.
We may assume $x = x^{A_i}$. Now $(Yx)(zx) = p_2$. Also $\mathbf{L}(Yx) \in \overline{\mathcal{O}_1}$. So $Yx \in \mathcal{L}_1$. We can not have $Yx \in \mathcal{Q}_1$ by Lemma 5.7. So $Yx = \mathbf{K}p_3$ with $p_3 \in \mathcal{P}_1$. $p_2 = (Yx)(zx) = p_3$. So $Yx = \mathbf{K}p_2$. Hence $\mathbf{L}(Yx) = \mathbf{H}p_2 \in \overline{\mathcal{O}}$ and the IH yields $\Gamma, Xx \vdash_{\mathcal{I}\Xi} \mathbf{H}p_2$. If $p = \mathbf{\Xi}A_i(\lambda x^{A_i}.p_2)$ then we have $\Gamma, Xx \vdash_{\mathcal{I}\Xi} \mathbf{H}((\lambda x^{A_i}.p_2)x^{A_i})$. So by rule Ξ_{H} we get $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{H}(\mathbf{\Xi}A_i(\lambda x^{A_i}.p_2))$, i.e. $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}(\mathbf{G}XY)$. If $p = \mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2)$ then $\Gamma, p_1 \vdash_{\mathcal{I}\Xi} \mathbf{H}p_2$ where $x \notin \mathrm{FV}(\Gamma, p_1)$, so $x \notin \mathrm{FV}(p_2)$ by Proposition B3.10. So $\Gamma, (\mathbf{K}p_1)x \vdash_{\mathcal{I}\Xi} \mathbf{H}((\mathbf{K}p_2)x)$ where $x \notin \mathrm{FV}(\mathbf{K}p_2)$, and therefore by rule Ξ_{H} we get $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{H}(\mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2))$, i.e. $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}(\mathbf{G}XY)$. □

5.24. Proposition (Completeness of $\mathcal{I}\mathbf{G}$ for PRED).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}} \varphi.$$

PROOF. This follows directly from the preceding proposition and Proposition B3.11. □

5.25. REMARK. In B4.1 we stated that it is possible to work with variants of the systems $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathbf{F}$ and $\mathcal{I}\mathbf{G}$ based on β-conversion only. This still holds. It might seem that in the proof of 5.9 we used the $\eta-$axiom, but that is not the case, because by Definition 5.2 not only $ft^{A_1} \in \mathcal{T}_1^{A_1}$, but also $f \in \mathcal{T}_1^{\mathbf{F}A_1A_1}$ and similarly $g \in \mathcal{T}_1^{\mathbf{F}A_1(\mathbf{F}A_2A_1)}$.

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