Completeness of some systems of illative combinatory logic for first-order propositional and predicate calculus[1]

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October 1995

Abstract

Illative combinatory logic consists of the theory of combinators or lambda calculus extended by extra constants (and corresponding axioms and rules) intended to capture inference. In a preceding paper, BBD[1993], we considered 4 systems of illative combinatory logic that are sound for first order propositional and predicate calculus. The interpretation from ordinary logic into the illative systems can be done in two ways: following the propositions-as-types paradigm, in which derivations become combinators, or in a more direct way, in which derivations are not translated. Both translations are closely related in a canonical way. In the cited paper we proved completeness of the two direct translations. In the present paper we prove that also the two indirect translations are complete. Moreover we shall prove that two of the systems are also complete for predicate calculus with higher type functions.

[1]Research supported by the Australian Research Council.

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1. Introduction

This paper is a continuation of a preceding paper, BBD[1993], by the same authors. We will refer to that paper simply as B; so for example Proposition B3.10 means Proposition 3.10 in BBD[1993]. As the present paper is a continuation of that paper, we will not repeat all the definitions and results here but will give only a very short summary.

In Definition B2.3 we defined 4 systems of illative combinatory logic $\mathcal{I}\mathbf{P}, \mathcal{I}\mathbf{\Xi}, \mathcal{I}\mathbf{F}$ and $\mathcal{I}\mathbf{G}$. We interpreted propositional and predicate calculus into illative systems in two ways: following the propositions-as-types paradigm, in which derivations become combinators, and in a more direct way, in which derivations are not translated. All translations turned out to be sound (Propositions B2.11 and B2.14). The 2 direct translations $[\ ]^1 : \mathrm{PROP} \to \mathcal{I}\mathbf{P}$ and $[\ ]^1 : \mathrm{PRED} \to \mathcal{I}\mathbf{\Xi}$ turned out to be complete (Propositions B3.11 and B3.14). We conjectured completeness for the interpretations $[\ ]^2 : \mathrm{PROP} \to \mathcal{I}\mathbf{F}$ and $[\ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathbf{G}$, but we were not able to prove that. These conjectures will be proved in the present paper.

The completeness proof for the direct translation $[\ ]^1 : \mathrm{PRED} \to \mathcal{I}\mathbf{\Xi}$ was given in the following way. We had to show

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

(c.f. page B781.) Write $\Gamma' = \Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi)$. First we defined a grammar $\mathcal{O}$ in order to analyze the terms M such that $\Gamma' \vdash_{\mathcal{I}\Xi} M$. We defined $\overline{\mathcal{O}} = \{M \mid \exists N{\in}\mathcal{O} \mid N =_{\beta\eta} M\}$ and it was clear that $\Gamma' \subseteq \overline{\mathcal{O}}$. We showed in Proposition B3.10 that $\overline{\mathcal{O}}$ is closed under derivations in $\mathcal{I}\mathbf{\Xi}$:

$$\Gamma \vdash_{\mathcal{I}\Xi} M, \Gamma \subseteq \overline{\mathcal{O}} \Rightarrow M{\in}\overline{\mathcal{O}}.$$

Using this result we then proved in Proposition B3.11

$$\Gamma' \vdash_{\mathcal{I}\Xi} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

The completeness proof for $[\ ]^1 : \mathrm{PROP} \to \mathcal{I}\mathbf{P}$ was done in a similar but easier way.

In the present paper we will prove completeness of $[\ ]^2 : \mathrm{PROP} \to \mathcal{I}\mathbf{F}$ and $[\ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathbf{G}$.

Completenes of $\mathcal{I}\mathbf{F}$ for PROP is relatively easy. Let us denote the grammar for the completeness proof of $\mathcal{I}\mathbf{P}$ for PROP (defined in B3.12) by $\mathcal{O}_0$ in order to distinguish it from $\mathcal{O}$ above. We shall prove in Proposition 2.3

$$\Gamma \vdash_{\mathcal{I}\mathrm{F}} M,\ \Gamma \subseteq \overline{\mathcal{O}_0} \ \Rightarrow\ \Gamma \vdash_{\mathcal{I}\mathrm{P}} M.$$

Then completenes of $\mathcal{I}\mathbf{F}$ for PROP follows directly from completenes of $\mathcal{I}\mathbf{P}$ for PROP.

Completeness of $\mathcal{I}\mathbf{G}$ for PRED is a little bit more complicated. Again we define a grammar, denoted by $\mathcal{O}_1$ to distinguish it from $\mathcal{O}$, and we will show in Proposition 3.11 that $\overline{\mathcal{O}_1}$ is closed under derivations in $\mathcal{I}\mathbf{G}$:

$$\Gamma \vdash_{\mathcal{I}\mathrm{G}} M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M \in \overline{\mathcal{O}_1}.$$

The grammar $\mathcal{O}_1$ is more complicated than $\mathcal{O}$ and we cannot prove completeness directly using this grammar, as in B3.11 for $\mathcal{I}\mathbf{\Xi}$ relative to PRED. But we shall use $\mathcal{O}_1$ in order to show in Lemma 3.19 that in a proof of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$, where $\Gamma \subseteq \overline{\mathcal{O}}$, all cut formulas can be eliminated. From this we get in Proposition 3.23 conservativity of $\mathcal{I}\mathbf{G}$ over $\mathcal{I}\mathbf{\Xi}$

$$\Gamma \subseteq \overline{\mathcal{O}}, M \in \overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M$$

and then completeness of $\mathcal{I}\mathbf{G}$ for PRED is an immediate corollary of this proposition and Proposition B3.11.

In the last part of the paper we shall introduce another logic system: Predicate Calculus with Higher Type Functions, notation PRED$\lambda\!\rightarrow$. We shall prove in Proposition 4.15 that $\mathcal{I}\mathbf{G}$ is also complete for PRED$\lambda\!\rightarrow$ under the second translation $[\ ]^2$. As a corollary we get completeness of $\mathcal{I}\mathbf{\Xi}$ for PRED$\lambda\!\rightarrow$ under the first translation $[\ ]^1$.

2. Completeness of $\mathcal{I}\mathbf{F}$ for PROP

We must show for $\varphi \in \mathbb{F}_{\mathrm{PROP}},\ \Delta \subseteq \mathbb{F}_{\mathrm{PROP}}$:

$$\forall r \exists M \in \Lambda [[\Delta]^2_r, \Gamma^2_r(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_r M] \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi.$$

It is sufficient to show

$$\exists M \in \Lambda [[\Delta]^2_{\mathsf{K}}, \Gamma^2_{\mathsf{K}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_{\mathsf{K}} M] \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi,$$

i.e. (by Proposition B2.15)

$$[\Delta]^1_{\mathsf{I}}, \Gamma^1_{\mathsf{I}}(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^1_{\mathsf{I}} \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi.$$

We omit I in the notation and we prove

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PROP}} \varphi. \qquad (*)$$

$(*)$ could be proved in exactly the same way as Proposition B3.14 (Completeness for $\mathcal{I}\mathbf{P}$ relative to PROP), changing only $\mathcal{I}\mathbf{P}$ in $\mathcal{I}\mathbf{F}$. The reason for this is the following. In the proof of Proposition B2.7 we showed that P_{e}, P_{i} and P_{H} follow from F_{e}, F_{i} and F_{L} by the substitutions of $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y. An inspection of the proof of completeness for $\mathcal{I}\mathbf{P}$ relative to PROP shows that every X and Y

occurring in the proof in cases P_e, P_i and P_H have the form $\mathbf{K}X_1$ and $\mathbf{K}Y_1$ and therefore also a substitution in the opposite direction is possible.

We will give another proof and show that (*) is a *consequence* of Proposition B3.14 by proving in Proposition 2.3 that we have

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{IF}} [\varphi]^1 \;\Rightarrow\; [\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{IP}} [\varphi]^1.$$

2.1. Definition. If $\mathcal{R}$ is any grammatical class in this paper then (c.f. Definition B3.2)

$$\overline{\mathcal{R}} = \{X \mid (\exists Y) Y \in \mathcal{R} \wedge X =_{\beta\eta} Y\}.$$

2.2. Definition (Grammar for derivable formulas for $\mathcal{IF}$). We copy Definition B3.12, adding the index 0.

$$\begin{array}{rcl} \mathcal{P}_0 & = & \mathcal{V}_0 \mid \mathcal{P}_0 \supset \mathcal{P}_0 \equiv \mathbf{\Xi}(\mathbf{K}\mathcal{P}_0)(\mathbf{K}\mathcal{P}_0); \\ \mathcal{G}_0 & = & \mathbf{H}\mathcal{P}_0 \mid \mathcal{P}_0 \supset \mathcal{G}_0 \equiv \mathbf{\Xi}(\mathbf{K}\mathcal{P}_0)(\mathbf{K}\mathcal{G}_0); \\ \mathcal{O}_0 & = & \mathcal{G}_0 \mid \mathcal{P}_0. \end{array}$$

Note that $[\Delta]^1, \Gamma^1(\Delta, \varphi) \subseteq \overline{\mathcal{O}_0}$.

2.3. Proposition. *$\mathcal{IF}$ is conservative over $\mathcal{IP}$ in the following sense*

$$\Gamma \vdash_{\mathcal{IF}} M \;\&\; \Gamma \subseteq \overline{\mathcal{O}_0} \;\Rightarrow\; \Gamma \vdash_{\mathcal{IP}} M \;\&\; M \in \overline{\mathcal{O}_0}. \qquad (**)$$

Proof. We rewrite the rules for $\mathcal{IF}$ in the following way:

$$\begin{array}{lllll} \mathsf{F}_\mathrm{e} & \Gamma \vdash \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash XV & \Rightarrow & \Gamma \vdash (Y \circ Z)V; \\ \mathsf{F}_\mathrm{i} & \Gamma, Xx \vdash (Y \circ Z)x,\ \Gamma \vdash \mathbf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y, Z) & \Rightarrow & \Gamma \vdash \mathbf{\Xi}X(Y \circ Z); \\ \mathsf{F}_\mathsf{L} & \Gamma, Xx \vdash \mathbf{L}Y,\ \Gamma \vdash \mathbf{L}X,\ x \notin \mathrm{FV}(\Gamma, X, Y) & \Rightarrow & \Gamma \vdash \mathbf{L}(\mathbf{F}XY). \end{array}$$

The proof of (**) is by induction on the proof of $\Gamma \vdash_{\mathcal{IF}} M$.

Case F_e. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} (Y \circ Z)V$ as consequence of
$\Gamma \vdash_{\mathcal{IF}} \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash_{\mathcal{IF}} XV$.

By the induction hypothesis one has

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{\Xi}X(Y \circ Z),\ \Gamma \vdash_{\mathcal{IP}} XV,$$

$$\mathbf{\Xi}X(Y \circ Z) = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O),\ XV = p,$$

where $p \in \mathcal{P}_0$ and $O \in \mathcal{O}_0$. By rule P_e we have $\Gamma \vdash_{\mathcal{IP}} O$, where $O = (Y \circ Z)V = M$.

Case $\mathsf{F_i}$. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} \mathbf{\Xi}X(Y \circ Z)$ as consequence of
$\Gamma \vdash_{\mathcal{IF}} \mathbf{L}X,\ \Gamma, Xx \vdash_{\mathcal{IF}} (Y \circ Z)x$ with $x \notin \mathrm{FV}(\Gamma, X, Y, Z)$.

By the IH we have

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{L}X,\ \ X = \mathbf{K}p,\ \ \Gamma, p \vdash_{\mathcal{IP}} (Y \circ Z)x$$

and $(Y \circ Z)x = O \in \mathcal{O}_0$. Now $x \notin \mathrm{FV}(\Gamma, p)$, so $x \notin \mathrm{FV}(O)$ by Proposition B3.10, hence $Y \circ Z = \mathbf{K}O$. Therefore by rule $\mathsf{P_i}$ we get

$$\Gamma \vdash_{\mathcal{IP}} p \supset O$$

and $p \supset O = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O) = \mathbf{\Xi}X(Y \circ Z) = M$.

Case $\mathsf{F_L}$. $\Gamma \vdash_{\mathcal{IF}} M$ is $\Gamma \vdash_{\mathcal{IF}} \mathbf{L}(\mathbf{F}XY)$ as consequence of
$\Gamma \vdash_{\mathcal{IF}} \mathbf{L}X,\ \Gamma, Xx \vdash_{\mathcal{IF}} \mathbf{L}(Y)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has

$$\Gamma \vdash_{\mathcal{IP}} \mathbf{L}X,\ \ X = \mathbf{K}p,\ \ \Gamma, p \vdash_{\mathcal{IP}} \mathbf{L}(Y),\ \ Y = \mathbf{K}p_1.$$

Hence by rule $\mathsf{P_H}$ $\Gamma \vdash_{\mathcal{IP}} \mathbf{H}(p \supset p_1)$. Now $\mathbf{H}(p \supset p_1) = \mathbf{L}(\mathbf{K}(p \supset p_1)) = \mathbf{L}(\mathbf{F}(\mathbf{K}p)(\mathbf{K}p_1)) = \mathbf{L}(\mathbf{F}XY)$. So $\Gamma \vdash_{\mathcal{IP}} M$. □

3. Completeness of $\mathcal{IG}$ for PRED

We will show for $\varphi \in \mathbb{F}_{\mathrm{PRED}},\ \Delta \subseteq \mathbb{F}_{\mathrm{PRED}}$:

$$\forall r[\ \Gamma^{2,+}_{r,s}, [\Delta]^2_r, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^2_r M \text{ some } M\ ] \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

It is sufficient to show

$$\Gamma^{2,+}_{\mathbf{K},s}, [\Delta]^2_{\mathbf{K}}, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^2_{\mathbf{K}} M \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi,$$

i.e.

$$\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

Here $\Gamma^{1,+}_s, [\Delta]^1$ and $[\varphi]^1$ stand for $\Gamma^{1,+}_{\mathbf{I},s}, [\Delta]^1_{\mathbf{I}}$ and $[\varphi]^1_{\mathbf{I}}$.

The proof is again in several steps. First we define in 3.2 a *grammar* in order to analyze the terms N such that $\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N$. Using this grammar we show in Lemma 3.19 that in a proof of $\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N$ all cuts can be eliminated . From this we get in Proposition 3.23

$$\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{IG}} N \ \Rightarrow\ \Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I\Xi}} N$$

and then completeness of $\mathcal{IG}$ for PRED is an immediate corollary of this proposition and Proposition B3.11. Instead of $\vdash_{\mathcal{IG}}$ we shall mostly write $\vdash$.

Now first we repeat Definition B3.2, the grammar for derivable statements in $\mathcal{I\Xi}$ for the manysorted structure of our example, in order to contrast it with the grammar for derivable statements in $\mathcal{IG}$ of Definition 3.2 and also because we need it in the sequel. Instead of $x_i, \mathcal{V}_i$ and $\mathcal{T}_i$ we shall write $x^{A_i}, \mathcal{V}^{A_i}$ and $\mathcal{T}^{A_i}$.

3.1. DEFINITION (Grammar for derivable statements in $\mathcal{I}\mathbf{\Xi}$).

$$\mathcal{T} = \mathcal{T}^{A_1} \mid \mathcal{T}^{A_2};$$
$$\mathcal{T}^{A_1} = \mathcal{V}^{A_1} \mid a \mid f\mathcal{T}^{A_1} \mid g\mathcal{T}^{A_1}\mathcal{T}^{A_2};$$
$$\mathcal{T}^{A_2} = \mathcal{V}^{A_2};$$

$$\mathcal{P} = P\mathcal{T}^{A_1} \mid \mathbf{\Xi}A_i(\lambda x^{A_i}.\mathcal{P}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P});$$
$$\mathcal{G} = \mathbf{L}A_i \mid A_i\mathcal{T}^{A_i} \mid \mathbf{\Xi}A_i(\lambda x^{A_i}.\mathcal{G}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}) \mid \mathbf{L}(\mathbf{K}\mathcal{P});$$
$$\mathcal{O} = \mathcal{G} \mid \mathcal{P}.$$

Now we will define the grammar $\mathcal{O}_1$ for derivable statements in $\mathcal{I}\mathbf{G}$. $\mathcal{O}_1$ is more complicated then the grammar $\mathcal{O}$ for $\mathcal{I}\mathbf{\Xi}$ for several reasons. Write Γ' for $\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi)$. If $\Gamma' \vdash \mathbf{L}X$ in system $\mathcal{I}\mathbf{\Xi}$ then $X = A_i$ or $X = \mathbf{K}p$ with $p \in \mathcal{P}$. But if $\Gamma' \vdash \mathbf{L}X$ in system $\mathcal{I}\mathbf{G}$ then we can have for example $X = \mathbf{F}A_1A_2$ or even $X = \mathbf{F}(\mathbf{K}p)A_2$. Therefore we define an extension $\mathcal{L}_1$ of the set of sorts $\{A_1, A_2\}$ containing these X's. We need for each $l \in \mathcal{L}_1$ a set of variables $\mathcal{V}_1^l$, similar to the sets $\mathcal{V}^{A_1}$ and $\mathcal{V}^{A_2}$. If we took a different set of variables for each $l \in \mathcal{L}_1$, then the substitution lemma 3.5, that is needed in the proof of Proposition 3.11, would not hold. Therefore we define for each $l \in \mathcal{L}_1$ the skeleton $\mathrm{sk}(l)$, such that $sk(\mathbf{K}p)$ is the same for all $p \in \mathcal{P}_1$ and we define $\mathcal{V}_1^{l_1} = \mathcal{V}_1^{l_2}$ iff $sk(l_1) = sk(l_2)$.

The sets $\mathcal{T}_1^l$ will contain $\lambda-$abstractions: if $l = \mathbf{F}l_1l_2$ and $t^{l_2} \in \mathcal{T}_1^{l_2}$ then $\lambda x^{l_1}.t^{l_2} \in \mathcal{T}_1^l$. This may seem a little bit strange, but in fact it is very natural. For example we have $\vdash \mathbf{F}A_1A_1(\lambda x^{A_1}.x^{A_1})$ by rule $\mathsf{F_i}$. Hence $\lambda x^{A_1}.x^{A_1}$ should be in $\mathcal{T}_1^{\mathbf{F}A_1A_1}$. Moreover in the proof of Lemma 3.9 we really need that $\lambda x^{l_1}.t^{l_2} \in \mathcal{T}_1^{\mathbf{F}l_1l_2}$.

Finally $\mathcal{P}_1, \mathcal{G}_1$ and $\mathcal{O}_1$ are very similar to $\mathcal{P}, \mathcal{G}$ and $\mathcal{O}$, except that besides $\mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O) \in \mathcal{O}_1$ we now also have $\mathbf{\Xi}(\mathbf{K}p)(\lambda x^{\mathbf{K}p}.O) \in \mathcal{O}_1$, where $x^{\mathbf{K}p}$ may occur free in O.

3.2. DEFINITION (Grammar for derivable statements in $\mathcal{I}\mathbf{G}$). (i) $\Lambda^s(\mathbf{\Xi}, \mathbf{L})$ is, as in Definition B2.12, $\Lambda(\mathbf{\Xi}, \mathbf{L})$ extended by the extra constants A_1, A_2, P, f, g, a associated with the signature s of the manysorted structure of our example.

(ii) Now we will define simultaneously sets $\mathcal{Q}_1$ and $\mathcal{L}_1$ of sorts (extensions of the set $\{A_1, A_2\}$), for each $l \in \mathcal{L}_1$ the skeleton $\mathrm{sk}(l)$, a set of variables $\mathcal{V}_1^l$ and a set of terms $\mathcal{T}_1^l$, sets $\mathcal{P}_1, \mathcal{G}_1$ and $\mathcal{O}_1$.

Definition of $\mathcal{Q}_1$ and $\mathcal{L}_1$.

$$\mathcal{Q}_1 = A_1 \mid A_2 \mid \mathbf{G}l(\lambda x^l.q) \quad (l \in \mathcal{L}_1, x^l \in \mathcal{V}_1^l, q \in \mathcal{Q}_1);$$
$$\mathcal{L}_1 = \mathcal{Q}_1 \cup \mathbf{K}\mathcal{P}_1.$$

Definition of $sk(l)$.

$$sk(A_i) = A_i;$$
$$sk(\mathbf{K}p) = \mathbf{K}(Px_1^{A_1}) \quad \text{(i.e. } sk(\mathbf{K}p) \text{ is independent of } p);$$
$$sk(\mathbf{G}l(\lambda x^l.q)) = \mathbf{G}(sk(l))(\lambda x^l.sk(q)).$$

Definition of $\mathcal{V}_1^l$.

We assume that for each skeleton $sk(l)$ there is a different set of variables

$$\mathcal{V}_1^{sk(l)} = \{x^{sk(l)}, x_1^{sk(l)}, x_2^{sk(l)}, \dots y^{sk(l)}, y_1^{sk(l)}, y_2^{sk(l)}, \dots\}$$

and for each $l \in \mathcal{L}_1$ we define

$$\mathcal{V}_1^l = \mathcal{V}_1^{sk(l)}, \quad x^l = x^{sk(l)}, x_i^l = x_i^{sk(l)}, \quad y^l = y^{sk(l)}, y_i^l = y_i^{sk(l)}.$$

Definition of $\mathcal{T}_1^l$.

$$\begin{array}{lll}
\mathcal{T}_1^l = a \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathsf{F}l_1 l}\mathcal{T}_1^{l_1}, & & \text{if } l \equiv A_1;\\
\mathcal{T}_1^l = \quad \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathsf{F}l_1 l}\mathcal{T}_1^{l_1}, & & \text{if } l \equiv A_2;\\
\mathcal{T}_1^l = f \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathsf{F}l_1 l}\mathcal{T}_1^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_1^{A_1}, & & \text{if } l \equiv \mathsf{F}A_1A_1;\\
\mathcal{T}_1^l = g \mid \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathsf{F}l_1 l}\mathcal{T}_1^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_1^{\mathsf{F}A_2A_1}, & & \text{if } l \equiv \mathsf{F}A_1(\mathsf{F}A_2A_1);\\
\mathcal{T}_1^l = \quad \mathcal{V}_1^l \mid \mathcal{T}_1^{\mathsf{G}l_1(\lambda x^{l_1}.l)}\mathcal{T}_1^{l_1} \mid \lambda x^{l_2}.\mathcal{T}_1^{l_3}, & & \text{if } l \equiv \mathsf{G}l_2(\lambda x^{l_2}.l_3), l \notin \{\mathsf{F}A_1A_1, \mathsf{F}A_1(\mathsf{F}A_2A_1)\};\\
\mathcal{T}_1^l = \Lambda^s(\mathsf{\Xi}, \mathsf{L}), & & \text{if } l \equiv \mathsf{K}p.
\end{array}$$

Definition of $\mathcal{P}_1, \mathcal{G}_1$ *and* $\mathcal{O}_1$.

$$\begin{array}{l}
\mathcal{P}_1 = P\mathcal{T}_1^{A_1} \mid \mathsf{\Xi}l(\lambda x^l.\mathcal{P}_1);\\
\mathcal{G}_1 = \mathsf{L}\mathcal{L}_1 \mid A_i\mathcal{T}_1^{A_i} \mid \mathsf{\Xi}l(\lambda x^l.\mathcal{G}_1);\\
\mathcal{O}_1 = \mathcal{P}_1 \mid \mathcal{G}_1.
\end{array}$$

(iii) *Notation.*

$$\begin{array}{ll}
q, q_1, q_2, q', \dots & \text{denote elements of } \mathcal{Q}_1;\\
l, l_1, l_2, l', \dots & \text{denote elements of } \mathcal{L}_1;\\
p, p_1, p_2, p', \dots & \text{denote elements of } \mathcal{P}_1.
\end{array}$$

3.3. Remarks. (i) If $\mathrm{FV}(q) = \emptyset$ then $\mathsf{G}l(\lambda x^l.q) =_\beta \mathsf{F}lq$. In that case we will denote $\mathsf{G}l(\lambda x^l.q)$ usually by $\mathsf{F}lq$. In fact we did that already in the above definition.

(ii) The grammar in the definition is not context-free and this causes some inaccuracies in the notation. In the definition of $\mathcal{P}_1$ (and similarly at some other places) instead of $\mathsf{\Xi}l(\lambda x^l.\mathcal{P}_1)$ we should write $\mathsf{\Xi}\mathcal{L}_1(\lambda x^{\mathcal{L}_1}.\mathcal{P}_1)$, but then we do not know that these two $\mathcal{L}_1$'s represent the *same* $l \in \mathcal{L}_1$. (We may not have $\mathsf{\Xi}l_1(\lambda x^{l_2}.\mathcal{P}_1)$.)

(iii) In the definition of $\mathcal{T}_1^l$ instead of the clause $\mathcal{T}_1^{\mathsf{G}l_1(\lambda x^{l_1}.l)}\mathcal{T}_1^{l_1}$ one might expect, in view of rule G_e a clause $t_1^{\mathsf{G}l_1(\lambda x^{l_1}.l')}t_2^{l_1}$ where $l = l'[x^{l_1} := t_2^{l_1}]$, but that does not make a difference because we have for $l \in \mathcal{L}_1$

$$\mathcal{T}_1^{l[x^{l_1} := t^{l_1}]} = \mathcal{T}_1^l,$$

as follows easily from

$$\mathcal{V}_1^{l[x^{l_1}:=t^{l_1}]} = \mathcal{V}_1^l$$

in Lemma 3.5.

3.4. Lemma. $l_1, l_2 \in \mathcal{L}_1 \;\Rightarrow\; \mathbf{G}l_1(\lambda x^{l_1}.l_2) \in \mathcal{L}_1$.

Proof. It is clear if $l_2 \in \mathcal{Q}_1$. Now let $l_2 = \mathbf{K}p$. Then
$\mathbf{G}l_1(\lambda x^{l_1}.l_2) = \lambda z.\mathbf{\Xi}l_1(\mathbf{S}(\lambda x^{l_1}.\mathbf{K}p)z) = \lambda z.\mathbf{\Xi}l_1(\lambda x^{l_1}.p) = \mathbf{K}(\mathbf{\Xi}l_1(\lambda x^{l_1}.p)) \in \mathbf{K}\mathcal{P}_1$. □

3.5. Lemma. *Let* $\mathcal{W} = \mathcal{Q}_1, \mathcal{L}_1, \mathcal{T}_1^l, \mathcal{P}_1, \mathcal{G}_1,$ *or* $\mathcal{O}_1$ *and let* $w \in \mathcal{W}, x^{l_1} \in \mathcal{V}_1^{l_1}, t^{l_1} \in \mathcal{T}_1^{l_1}$, *then*

$$w[x^{l_1} := t^{l_1}] \in \mathcal{W},$$

moreover if $\mathcal{W} = \mathcal{L}_1$ *then*

$$sk(w[x^{l_1} := t^{l_1}]) = sk(w).$$

and hence

$$\mathcal{V}_1^{w[x^{l_1}:=t^{l_1}]} = \mathcal{V}_1^w.$$

Proof. By induction on the structure of w. Instead of x^{l_1} and t^{l_1} we shall write x and t.

Case $w \in \mathcal{Q}_1$.

$w = A_i$. This is trivally ok.

$w = \mathbf{G}l_2(\lambda y^{l_2}.q)$. Then $w[x := t] = \mathbf{G}(l_2[x := t])(\lambda y^{l_2}.q[x := t]) =$
$= \mathbf{G}(l_2[x := t])(\lambda y^{l_2[x:=t]}.q[x := t])$
(by the second part of the IH for l_2) and this is in $\mathcal{Q}_1$ by the first part of the IH for l_2 and q. The second part of the IH for l_2 and q yields $sk(w[x := t]) = sk(w)$.

Case $w \in \mathcal{L}_1$.
This follows directly from the cases $w \in \mathcal{Q}_1, w \in \mathcal{P}_1$.

Case $w = t^l \in \mathcal{T}_1^l$.

$l = A_1$. Now $w = x_1^{A_1}, a$ or $t_1^{\mathbf{G}l(\lambda x_1^l.A_1)}t_2^l$. All these cases are easy.

$l = A_2$. Easy.

$l = \mathbf{G}l_2(\lambda y^{l_2}.q)$. Now $w = x_1^l, t_1^{\mathbf{G}l_3(\lambda x_1^{l_3}.l)}t_2^{l_3}, \lambda y^{l_2}.t_1^q, f$ or g. Easy again.

$l = \mathbf{K}p$. Then $w[x := t] \in \Lambda = \mathcal{T}_1^{\mathbf{K}p}$.

Case $w{\in}\mathcal{P}_1$.

$w = Pt_1^{A_1}$. Then $w[x := t] = P(t_1^{A_1}[x := t])$.

$w = \mathbf{\Xi}l_2(\lambda y^{l_2}.p)$. Then $w[x := t] = \mathbf{\Xi}(l_2[x := t])(\lambda y^{l_2}.p[x := t]) =$
$= \mathbf{\Xi}(l_2[x := t])(\lambda y^{l_2[x:=t]}.p[x := t])$.

Case $w{\in}\mathcal{G}_1$. Similarly. □

3.6. Lemma. *Let* $cX_1 \dots X_n =_{\beta\eta} M$ *for some constant* c *and some* $M{\in}\mathcal{O}_1$. *Then* $n{\in}\{1,2\}$ *and* $M \equiv cY_1 \dots Y_n$ *with* $Y_i =_{\beta\eta} X_i$.

Proof. By Church-Rosser for $\beta\eta$-reduction. □

3.7. Lemma. *Let* $q{\in}\mathcal{Q}_1$ $p{\in}\mathcal{P}_1$, $M{\in}\Lambda^s(\mathbf{\Xi},\mathbf{L})$ *and let* σ *substitute variables for variables. Then* $qM \neq_{\beta\eta} p^\sigma$.

Proof. By an easy induction on q, using Church-Rosser. □

In the definition of $\mathcal{G}_1$ one of the clauses is $A_i\mathcal{T}_1^{A_i}$ where one might expect the more general clause $q\mathcal{T}_1^q$. But $A_i\mathcal{T}_1^{A_i}$ is sufficient as follows from the next lemma.

3.8. Lemma.
$$qt^q{\in}\overline{\mathcal{G}_1} \text{ for each } q{\in}\mathcal{Q}_1.$$

Proof. By induction on the structure of q. It is clear if $q = A_1$ or A_2. So let $q = \mathbf{G}l_1(\lambda x^{l_1}.q_1)$. Then $qt^q = \mathbf{G}l_1(\lambda x^{l_1}.q_1)t^q = \mathbf{\Xi}l_1(\mathbf{S}(\lambda x^{l_1}.q_1)t^q) = \mathbf{\Xi}l_1(\lambda x^{l_1}.q_1(t^q x^{l_1}))$. Now $t^q \in \mathcal{T}_1^q = \mathcal{T}_1^{\mathbf{G}l_1(\lambda x^{l_1}.q_1)}$ and $x^{l_1}{\in}\mathcal{T}_1^{l_1}$ hence $t^q x^{l_1}{\in}\mathcal{T}_1^{q_1}$. So by the induction hypothesis one has $q_1(t^q x^{l_1}){\in}\overline{\mathcal{G}_1}$. Hence $\mathbf{\Xi}l_1(\lambda x^{l_1}.q_1(t^q x^{l_1})){\in}\overline{\mathcal{G}_1}$. □

3.9. Lemma.
$$lV{\in}\overline{\mathcal{O}_1} \;\Rightarrow\; V{\in}\overline{\mathcal{T}_1^l}.$$

Proof. By induction on the structure of l. It is clear if $l = A_1, A_2$ or $\mathbf{K}p$. So let $l = \mathbf{G}l_1(\lambda x^{l_1}.q)$. Then $lV = \mathbf{\Xi}l_1(\lambda x^{l_1}.q(Vx^{l_1}))$. So $q(Vx^{l_1}){\in}\overline{\mathcal{O}_1}$ and hence by the induction hypothesis we have $Vx^{l_1} = t^q{\in}\mathcal{T}_1^q$. So $V = \lambda x^{l_1}.t^q{\in}\mathcal{T}_1^l$. □

3.10. Remark. Clearly $(\mathbf{K}p)t^{\mathbf{K}p} = p{\in}\mathcal{P}_1$, hence combining Lemma's 3.8 and 3.9 we get
$$lV{\in}\overline{\mathcal{O}_1} \;\Leftrightarrow\; V{\in}\overline{\mathcal{T}_1^l}.$$

Now we will prove $\Gamma \vdash M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M{\in}\overline{\mathcal{O}_1}$. The proof is different from the proof of Proposition B3.10, because we do not have
$$\Gamma \vdash M,\ \Gamma \subseteq \overline{\mathcal{O}_1}\ \&\ u{\notin}_{\beta\eta}\mathrm{FV}(\Gamma) \;\Rightarrow\; u{\notin}_{\beta\eta}\mathrm{FV}(M).$$
This is shown by the following example
$$\mathbf{F}(\mathbf{K}p)A_1x,\ p \vdash A_1(xu).$$
In fact this is the reason that in the grammar $\mathcal{O}_1$ we have elements $\mathbf{\Xi}(\mathbf{K}p)(\lambda x^{\mathbf{K}p}.O)$ instead of the simpler elements $\mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O)$.

3.11. PROPOSITION. *In* $\mathcal{I}\mathsf{G}$ *we have*

$$\Gamma \vdash M, \Gamma \subseteq \overline{\mathcal{O}_1} \;\Rightarrow\; M \in \overline{\mathcal{O}_1}.$$

PROOF. By induction on the derivation of $\Gamma \vdash M$. We only consider the three specific rules $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$ and G_{L}. (The two general rules are easy and $\mathsf{G}_{\mathrm{e}}, \mathsf{G}_{\mathrm{i}}$ are equivalent to $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$.)

Case Ξ_{e}. $\Gamma \vdash M$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \mathbf{\Xi}XY$, $\Gamma \vdash XV$.

By the IH one has $\mathbf{\Xi}XY \in \overline{\mathcal{O}_1}, XV \in \overline{\mathcal{O}_1}$. From $\mathbf{\Xi}XY \in \overline{\mathcal{O}_1}$ it follows that $X = l, Y = \lambda x^l.O$ where $O \in \mathcal{O}_1$. Now $XV = lV \in \overline{\mathcal{O}_1}$, hence $V = t^l \in \mathcal{T}_1^l$ by Lemma 3.9. $M = (\lambda x^l.O)t^l = O[x^l := t^l] \in \mathcal{O}_1$ by Lemma 3.5.

Case Ξ_{i}. $\Gamma \vdash M$ is $\Gamma \vdash \mathbf{\Xi}XY$ as direct consequence of
$\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash Yx$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has $\mathbf{L}X \in \overline{\mathcal{O}_1}$, hence $X = l \in \mathcal{L}_1$. Now x is any variable, so we may assume that $x = x^l \in \mathcal{V}_1^l$. Then $lx \in \overline{\mathcal{O}_1}$ by Lemma 3.8, hence by the IH we have $Yx \in \overline{\mathcal{O}_1}$. Let $Yx = O \in \overline{\mathcal{O}_1}$. Then $M = \mathbf{\Xi}l(\lambda x^l.O) \in \overline{\mathcal{O}_1}$.

Case G_{L}. $\Gamma \vdash M$ is $\Gamma \vdash \mathbf{L}(\mathbf{G}XY)$ as direct consequence of
$\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash \mathbf{L}(Yx)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

By the IH one has $\mathbf{L}X \in \overline{\mathcal{O}_1}$, hence $X = l \in \mathcal{L}_1$. Again we may assume that $x = x^l \in \mathcal{V}_1^l$. Then $lx \in \overline{\mathcal{O}_1}$, hence by the IH we have $\mathbf{L}(Yx) \in \overline{\mathcal{O}_1}$. So $Yx = l_1 \in \mathcal{L}_1$. $\mathbf{G}XY = \mathbf{G}l(\lambda x^l.l_1)) \in \overline{\mathcal{L}_1}$ by Lemma 3.4. Hence $\mathbf{L}(\mathbf{G}XY) \in \overline{\mathcal{G}_1} \subseteq \overline{\mathcal{O}_1}$. □

The grammar $\mathcal{O}_1$ is more complicated than $\mathcal{O}$ and we cannot prove completeness directly using this grammar, as in in B3.11 for $\mathcal{I}\mathbf{\Xi}$ relative to PRED. We will show in 3.12-3.21 that in each proof $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$, where $\Gamma \subseteq \overline{\mathcal{O}_1}$, all cut formulas can be eliminated. From this we get in Proposition 3.23 conservativity of $\mathcal{I}\mathsf{G}$ over $\mathcal{I}\mathbf{\Xi}$

$$\Gamma \subseteq \overline{\mathcal{O}}, M \in \overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M$$

and then completeness of $\mathcal{I}\mathsf{G}$ for PRED is an immediate corollary of this proposition and Proposition B3.11.

Cut elimination

First we define the $\mathbf{\Xi}-$length $L_{\mathbf{\Xi}}(O)$ for $O \in \overline{\mathcal{O}_1}$ (and $L_{\mathbf{\Xi}}(l)$ for $l \in \mathcal{L}_1$) and state some lemma's, without the (trivial) proofs.

3.12. DEFINITION ($\mathbf{\Xi}-$length $L_{\mathbf{\Xi}}$).

$$L_{\mathbf{\Xi}}(Pt^{A_1}) = L_{\mathbf{\Xi}}(\mathbf{L}l) = L_{\mathbf{\Xi}}(A_i t^{A_i}) = 0, \;\; L_{\mathbf{\Xi}}(\mathbf{\Xi}l(\lambda x^l.O)) = 1 + L_{\mathbf{\Xi}}(l) + L_{\mathbf{\Xi}}(O),$$

$$L_{\mathbf{\Xi}}(A_i) = 0, \;\; L_{\mathbf{\Xi}}(\mathbf{G}l(\lambda x^l.q)) = 1 + L_{\mathbf{\Xi}}(l) + L_{\mathbf{\Xi}}(q), \;\; L_{\mathbf{\Xi}}(\mathbf{K}p) = L_{\mathbf{\Xi}}(p).$$

3.13. Lemma. (i) $X,Y\in\mathcal{L}_1\cup\mathcal{O}_1,\ X\twoheadrightarrow_{\beta\eta}Y\ \Rightarrow\ L_\Xi(X)=L_\Xi(Y)$.
(ii) $X,Y\in\mathcal{L}_1\cup\mathcal{O}_1,\ X=_{\beta\eta}Y\ \Rightarrow\ L_\Xi(X)=L_\Xi(Y)$.

3.14. Definition. Let $M\in\overline{\mathcal{O}_1},\ M=_{\beta\eta}O$. Then $L_\Xi(M)=L_\Xi(O)$.

3.15. Lemma. $L_\Xi(lt^l)=L_\Xi(l)$.

3.16. Lemma. $X\in\mathcal{L}_1\cup\overline{\mathcal{O}_1},\ t^l\in\mathcal{T}_1^l\ \Rightarrow\ L_\Xi(X[x^l:=t^l])=L_\Xi(X)$.

3.17. Lemma.

$\Xi XY\in\overline{\mathcal{O}_1},\ XV\in\overline{\mathcal{O}_1}\ \Rightarrow\ L_\Xi(XV)<L_\Xi(\Xi XY)\ \&\ L_\Xi(YV)<L_\Xi(\Xi XY)$.

Proof. $\Xi XY=\Xi l(\lambda x^l.O)$ and $V\in\overline{\mathcal{T}_1^l}$. Now use Lemma's 3.15 and 3.16.

3.18. Definition. A cut elimination is a proof reduction of the following form

$$\dfrac{\dfrac{\boxed{D_1}\quad\overset{\overset{x}{X}}{\boxed{D_2}}}{\mathbf{L}X\quad Y}\qquad\boxed{D_3}}{\dfrac{\Xi XY\qquad XV}{YV}}\qquad\Longrightarrow\qquad\dfrac{\boxed{D_3}}{\dfrac{XV}{\boxed{[V/x]D_2}}}\ YV$$

ΞXY is called the cut formula.

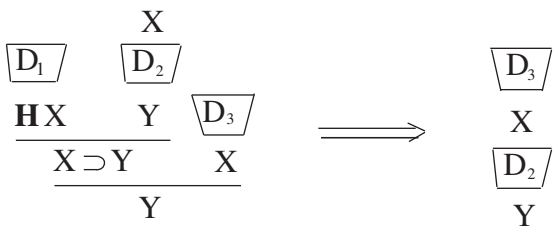
3.19. Lemma. *In a proof of* $\Gamma\vdash_{\mathcal{I}G}M$ *(or* $\Gamma\vdash_{\mathcal{I}\Xi}M$ *), where* $\Gamma\subseteq\overline{\mathcal{O}_1}$, *all cut formulas can be eliminated by the above proof reduction.*

Proof. By induction on the maximal Ξ−length of cut formulas in the proof of $\Gamma\vdash_{\mathcal{I}G}M$. Consider a cut in the proof with a cut formula ΞXY of maximal Ξ−length. Eliminate this cut. Now new cut formulas may have been introduced at XV and YV, however by Lemma 3.17 these have shorter Ξ−length. In the same way all other cut formulas of maximal Ξ−length can be eliminated and so by the induction hypothesis all cut formulas can be eliminated. □

3.20. Remark. The above can easily be generalised to allow $\beta\eta$−equality steps (Eq-steps) as follows.

$$\dfrac{\dfrac{\dfrac{\boxed{D_1}\quad\overset{\overset{x}{X}}{\boxed{D_2}}}{\mathbf{L}X\quad Y}}{\mathrm{Eq}\dfrac{\Xi XY}{\Xi X'Y'}}\qquad\boxed{D_3}}{Y'V}\qquad\Longrightarrow\qquad\dfrac{\dfrac{\boxed{D_3}}{\mathrm{Eq}\dfrac{X'V}{XV}}}{\dfrac{\boxed{[V/x]D_2}}{YV}}\ \mathrm{Eq}\dfrac{}{Y'V}$$

3.21. Remark. Similarly we can define cut elimination for $\mathcal{I}\mathbf{P}$ as a proof reduction of the following form.



This is in fact a special case of Definition 3.18. (Substitute $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y in Definition 3.18.) Also in a proof of $\Gamma \vdash_{\mathcal{I}\mathrm{P}} M$ where $\Gamma \subseteq \overline{\mathcal{O}_1}$ all cut formulas can be eliminated.

Now we need only one more lemma to derive conservativity of $\mathcal{I}\mathbf{G}$ over $\mathcal{I}\mathbf{\Xi}$ in Proposition 3.23.

3.22. Lemma. *Let $\Gamma \subseteq \overline{\mathcal{O}}$ and let $\Gamma \vdash_{\mathcal{I}\mathrm{G}} N$ be a cut free proof. If the last applied rule in the proof is not the $\Xi_{\mathrm{i}}-$rule, nor the $\mathsf{G}_{\mathsf{L}}-$rule (modulo the $\beta\eta-$rule), then $N \in \overline{\mathcal{O}}$.*

Proof. The proof is by induction on $\Gamma \vdash N$. The two general cases are easy, so we only consider

Case Ξ_{e}. $\Gamma \vdash N$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \mathbf{\Xi}XY$, $\Gamma \vdash XV$.

By the induction hypothesis for $\Gamma \vdash \mathbf{\Xi}XY$ we have $\mathbf{\Xi}XY \in \overline{\mathcal{O}}$. We distinguish two subcases

Subcase a). $\mathbf{\Xi}XY = \mathbf{\Xi}A_i(\lambda x^{A_i}.O)$ with $O \in \overline{\mathcal{O}}$. The induction hypothesis for $\Gamma \vdash A_iV$ yields $A_iV \in \overline{\mathcal{O}}$ and hence $V \in \overline{\mathcal{T}^{A_i}}$. So $N = YV = O[x^{A_i} := V] \in \overline{\mathcal{O}}$.

Subcase b). $\mathbf{\Xi}XY = \mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O)$ with $O \in \overline{\mathcal{O}}$. Now $YV = O \in \overline{\mathcal{O}}$. □

3.23. Proposition. *$\mathcal{I}\mathbf{G}$ is conservative over $\mathcal{I}\mathbf{\Xi}$ in the following sense*

$$\Gamma \subseteq \overline{\mathcal{O}}, M \in \overline{\mathcal{O}}, \Gamma \vdash_{\mathcal{I}\mathrm{G}} M \;\Rightarrow\; \Gamma \vdash_{\mathcal{I}\Xi} M.$$

Proof. By induction on the derivation of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$. We may assume that the proof is cut free. Again we only consider the three specific rules $\Xi_{\mathrm{e}}, \Xi_{\mathrm{i}}$ and G_{L}.

Case Ξ_{e}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} YV$ as direct consequence of
$\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{\Xi}XY$, $\Gamma \vdash_{\mathcal{I}\mathrm{G}} XV$.

$\mathbf{\Xi}XY \in \overline{\mathcal{O}}$ by Lemma 3.22, so by the IH we have $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{\Xi}XY$. Now $X = A_i$ or $X = \mathbf{K}p_1$, so also $XV \in \overline{\mathcal{O}}$ (use Lemma 3.22 for the case: $X = A_i$). Again by the

IH we get $\Gamma \vdash_{\mathcal{I}\Xi} XV$. So $\Gamma \vdash_{\mathcal{I}\Xi} YV$ by rule Ξ_{e}.

Case Ξ_{i}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{\Xi}XY$ as direct consequence of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{I}\mathrm{G}} Yx$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

$\mathbf{\Xi}XY \in \overline{\mathcal{O}}$ so $X = A_i$ or $X = \mathbf{K}p_1$ and $Yx = O \in \overline{\mathcal{O}}$. So $\mathbf{L}X \in \overline{\mathcal{O}}$, hence we have by the IH $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}X$. Now $\Gamma, Xx \subseteq \overline{\mathcal{O}}, Yx \in \overline{\mathcal{O}}$, so $\Gamma, Xx \vdash_{\mathcal{I}\Xi} Yx$ by the IH. Hence $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{\Xi}XY$.

Case G_{L}. $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ is $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}(\mathbf{G}XY)$ as direct consequence of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}X$, $\Gamma, Xx \vdash_{\mathcal{I}\mathrm{G}} \mathbf{L}(Yx)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

$\mathbf{L}(\mathbf{G}XY) \in \overline{\mathcal{O}}$ so $\mathbf{G}XY = \mathbf{K}p$, where $p = \mathbf{\Xi}A_i(\lambda x^{A_i}.p_2)$ or $p = \mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2)$ with $p_1, p_2 \in \mathcal{P}$. Since $\mathbf{L}X \in \overline{\mathcal{O}}$, we have by the IH $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}X$.
We may assume $x = x^{A_i}$. Now $(Yx)(zx) = p_2$. Also $\mathbf{L}(Yx) \in \overline{\mathcal{O}_1}$. So $Yx \in \mathcal{L}_1$. We can not have $Yx \in \mathcal{Q}_1$ by Lemma 3.7. So $Yx = \mathbf{K}p_3$ with $p_3 \in \mathcal{P}_1$. $p_2 = (Yx)(zx) = p_3$. So $Yx = \mathbf{K}p_2$. Hence $\mathbf{L}(Yx) = \mathbf{H}p_2 \in \overline{\mathcal{O}}$ and the IH yields $\Gamma, Xx \vdash_{\mathcal{I}\Xi} \mathbf{H}p_2$. If $p = \mathbf{\Xi}A_i(\lambda x^{A_i}.p_2)$ then we have $\Gamma, Xx \vdash_{\mathcal{I}\Xi} \mathbf{H}((\lambda x^{A_i}.p_2)x^{A_i})$. So by rule Ξ_{H} we get $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{H}(\mathbf{\Xi}A_i(\lambda x^{A_i}.p_2))$, i.e. $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}(\mathbf{G}XY)$. If $p = \mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2)$ then $\Gamma, p_1 \vdash_{\mathcal{I}\Xi} \mathbf{H}p_2$ where $x \notin \mathrm{FV}(\Gamma, p_1)$, so $x \notin \mathrm{FV}(p_2)$ by Proposition B3.10. So $\Gamma, (\mathbf{K}p_1)x \vdash_{\mathcal{I}\Xi} \mathbf{H}((\mathbf{K}p_2)x)$ where $x \notin \mathrm{FV}(\mathbf{K}p_2)$, and therefore by rule Ξ_{H} we get $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{H}(\mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2))$, i.e. $\Gamma \vdash_{\mathcal{I}\Xi} \mathbf{L}(\mathbf{G}XY)$. □

3.24. Proposition (Completeness of $\mathcal{I}\mathbf{G}$ for PRED).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}} \varphi.$$

Proof. This follows directly from the preceding proposition and Proposition B3.11. □

4. Completeness of $\mathcal{I}\mathbf{\Xi}$ and $\mathcal{I}\mathbf{G}$ for PRED$\lambda\!\rightarrow$

In this section we will show completeness of $\mathcal{I}\mathbf{\Xi}$ and $\mathcal{I}\mathbf{G}$ over predicate logic extended with higher type functions. In this logic we may quantify over all functions based on $\mathbf{A}_1$ and $\mathbf{A}_2$ and also over functions ranging over functions etc. and we can state e.g.

$$\forall f_1 \in ((\mathbf{A}_1 \rightarrow \mathbf{A}_2) \rightarrow \mathbf{A}_1) \forall f_2 \in (\mathbf{A}_1 \rightarrow \mathbf{A}_2)[\mathbf{P}(f_1 f_2)].$$

4.1. Definition. PRED$\lambda\!\rightarrow_0$, a preliminary version of Predicate Calculus with Higher Type Functions, is defined by

(i) *Types of* PRED$\lambda\!\rightarrow_0$

$$\mathbb{L}_{\mathrm{PRED}\lambda\rightarrow_0} = \mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbb{L}_{\mathrm{PRED}\lambda\rightarrow_0} \rightarrow \mathbb{L}_{\mathrm{PRED}\lambda\rightarrow_0}.$$

(ii) *Variables of* PRED$\lambda\to_0$

$$\mathbb{V}^{\mathbf{l}} = \{x_1^{\mathbf{l}}, x_2^{\mathbf{l}}, \ldots\} \quad \text{for } \mathbf{l}\in\mathbb{L}_{\text{PRED}\lambda\to_0}.$$

(iii) *Terms of* PRED$\lambda\to_0$

$$\begin{aligned}
\mathbb{T}^{\mathbf{l}} &= \mathbf{a} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\to\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, && \text{if } \mathbf{l} = \mathbf{A}_1;\\
\mathbb{T}^{\mathbf{l}} &= \quad \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\to\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, && \text{if } \mathbf{l} = \mathbf{A}_2;\\
\mathbb{T}^{\mathbf{l}} &= \mathbf{f} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\to\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, && \text{if } \mathbf{l} = \mathbf{A}_1\to\mathbf{A}_1;\\
\mathbb{T}^{\mathbf{l}} &= \mathbf{g} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\to\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, && \text{if } \mathbf{l} = \mathbf{A}_1\to(\mathbf{A}_2\to\mathbf{A}_1);\\
\mathbb{T}^{\mathbf{l}} &= \quad \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\to\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, && \text{if } \mathbf{l}\notin\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_1\to\mathbf{A}_1, \mathbf{A}_1\to(\mathbf{A}_2\to\mathbf{A}_1)\}.
\end{aligned}$$

(iv) *Well formed formulas of* PRED$\lambda\to_0$

$$\mathbb{F}_{\text{PRED}\lambda\to_0} = \mathbf{P}\mathbb{T}^{\mathbf{A}_1} \mid \mathbb{F}_{\text{PRED}\lambda\to_0} \supset \mathbb{F}_{\text{PRED}\lambda\to_0} \mid (\forall x^{\mathbf{l}})\mathbb{F}_{\text{PRED}\lambda\to_0}.$$

(v) *Postulates of* PRED$\lambda\to_0$

$$\begin{aligned}
\varphi\in\Gamma &\quad\Rightarrow\quad \Gamma\vdash\varphi;\\
\Gamma\vdash\varphi\supset\psi,\ \Gamma\vdash\varphi &\quad\Rightarrow\quad \Gamma\vdash\psi;\\
\Gamma,\varphi\vdash\psi &\quad\Rightarrow\quad \Gamma\vdash\varphi\supset\psi;\\
\Gamma\vdash\forall x^{\mathbf{l}}\varphi,\ t\in\mathbb{T}^{\mathbf{l}} &\quad\Rightarrow\quad \Gamma\vdash\varphi[x^{\mathbf{l}} := t];\\
\Gamma\vdash\varphi,\ x^{\mathbf{l}}\notin\text{FV}(\Gamma) &\quad\Rightarrow\quad \Gamma\vdash\forall x^{\mathbf{l}}\varphi.
\end{aligned}$$

4.2. Definition. The mappings $[-]^i_r$ and Γ in B2.12 from $\mathbb{F}_{\text{PRED}}$ into $\Lambda^s(\boldsymbol{\Xi}, \mathbf{L})$ and illative contexts respectively are extended to mappings from $\mathbb{F}_{\text{PRED}\lambda\to_0}$ by altering some clauses and adding some clauses in the following way.

First we define $[\mathbf{A}_j]^i_r = A_j$ $(j = 1, 2)$, $[\mathbf{l}_1\to\mathbf{l}_2]^i_r = \mathbf{F}[\mathbf{l}_1]^i_r[\mathbf{l}_2]^i_r$. We will denote $[\mathbf{l}]^i_r$ by l. Then we define

$[x^{\mathbf{l}}]^i_r = x^l$, $[\mathbf{t}_1\mathbf{t}_2]^i_r = [\mathbf{t}_1]^i_r[\mathbf{t}_2]^i_r$, $[\forall x^{\mathbf{l}}\varphi]^1_r = \boldsymbol{\Xi}l(\lambda x^l.[\varphi]^1_r), [\forall x^{\mathbf{l}}\varphi]^2_r = \mathbf{G}l(\lambda x^l.[\varphi]^2_r)$,
$\Gamma(x^{\mathbf{l}}) = lx^l$, $\Gamma(\forall x^{\mathbf{l}}\varphi) = \Gamma(\varphi) - \{lx^l\}$, $\Gamma(\mathbf{t}_1\mathbf{t}_2) = \Gamma(\mathbf{t}_1)\cup\Gamma(\mathbf{t}_2)$,
$\Gamma^1_{r,s} =< \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1A_1f, \mathbf{F}A_1(\mathbf{F}A_2A_1)g, \mathbf{F}A_1\mathbf{H}(r\circ P), A_1a >$,
$\Gamma^2_{r,s} =< \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1A_1f, \mathbf{F}A_1(\mathbf{F}A_2A_1)g, \mathbf{F}A_1\mathbf{L}(r\circ P), A_1a >$,
$\Gamma^{i,+}_{r,s} = \Gamma^i_{r,s}\cup\{lx^l_1 \mid l\neq A_1\}$,
$[-]^i, \Gamma^i_s$ and $\Gamma^{i,+}_s$ are obtained from $[-]^i_r, \Gamma^i_{r,s}$ and $\Gamma^{i,+}_{r,s}$ by everywhere omitting r i.e. by replacing $r\circ P$ by P.

4.3. PROPOSITION. *$\mathcal{I}\mathsf{G}$ is not complete for* PRED$\lambda\!\rightarrow_0$.

PROOF. Let $\varphi = \forall x_1^{\mathbf{A}_1 \to \mathbf{A}_1} \forall x_2^{\mathbf{A}_1} P(x_1 x_2) \supset \forall x_2^{\mathbf{A}_1} P x_2$. We will show

$$\forall r[\Gamma^{2,+}_{r,s}, \Gamma(\varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^2_r M \text{ some } M], \tag{$*$}$$

$$\nVdash_{\mathrm{PRED}\lambda\rightarrow_0} \varphi. \tag{$**$}$$

Proof of $(*)$. Let $X_1 = \mathsf{G}(\mathsf{F}A_1A_1)(\lambda x_1.\mathsf{G}A_1(\lambda x_2.(r\circ P)(x_1x_2)))$,
$Y_1 = \mathsf{G}A_1(\lambda x_2.(r\circ P)x_2), M = \lambda x^{X_1}.x^{X_1}I$ where $I = \lambda x^{A_1}.x^{A_1}$. We will prove $(*)$ by showing $\forall r[\Gamma^{2,+}_{r,s} \vdash_{\mathcal{I}\mathrm{G}} \mathsf{F}X_1Y_1M]$.

Note that the rules G_e and G_i imply the rules F_e and F_i. Let r be a closed term in $\Lambda^s(\mathsf{\Xi}, \mathsf{L})$. We have by rule F_i

$$\mathsf{L}A_1 \vdash \mathsf{F}A_1A_1I$$

and from this we get by rule G_e (taking
$X = \mathsf{F}A_1A_1, Y = \lambda x_1.\mathsf{G}A_1(\lambda x_2.(r\circ P)(x_1x_2)), Z = x(= x^{X_1}), V = I$)

$$\mathsf{L}A_1, X_1x \vdash Y_1(xI).$$

So we have

$$\mathsf{L}A_1, X_1x \vdash Y_1(Mx). \tag{1}$$

Moreover rule F_e yields

$$\mathsf{L}A_1, \mathsf{F}A_1A_1x_1, A_1x_2 \vdash A_1(x_1x_2).$$

Hence we get by applying rule F_e once more

$$\mathsf{L}A_1, \mathsf{F}A_1\mathsf{L}(r\circ P), \mathsf{F}A_1A_1x_1, A_1x_2 \vdash \mathsf{L}((r\circ P)(x_1x_2))$$

and by rule G_L twice

$$\mathsf{L}A_1, \mathsf{F}A_1\mathsf{L}(r\circ P) \vdash \mathsf{L}X_1. \tag{2}$$

From (1) and (2) we get by rule F_i

$$\mathsf{L}A_1, \mathsf{F}A_1\mathsf{L}(r\circ P) \vdash \mathsf{F}X_1Y_1M.$$

This proves $(*)$. But $\nVdash_{\mathrm{PRED}\lambda\rightarrow_0} \varphi$ because PRED$\lambda\!\rightarrow_0$ has no identity function in $\mathbb{T}^{\mathsf{F}\mathbf{A}_1\mathbf{A}_1}$. □

By similar examples one shows the need for other combinators or λ−terms. In fact all of these are needed in PRED$\lambda\!\rightarrow_0$ before completeness can be proved.

4.4. Definition (Predicate Calculus with Higher Type Functions, PRED$\lambda\!\rightarrow$).
Types and variables are as for PRED$\lambda\!\rightarrow_0$.
Terms of PRED$\lambda\!\rightarrow$

$$\begin{array}{lll}
\mathbb{T}^{\mathbf{l}} = \mathbf{a} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\rightarrow\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, & & \text{if } \mathbf{l}=\mathbf{A}_1;\\
\mathbb{T}^{\mathbf{l}} = \quad \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\rightarrow\mathbf{l}}\mathbb{T}^{\mathbf{l}_1}, & & \text{if } \mathbf{l}=\mathbf{A}_2;\\
\mathbb{T}^{\mathbf{l}} = \mathbf{f} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\rightarrow\mathbf{l}}\mathbb{T}^{\mathbf{l}_1} \mid \lambda x^{\mathbf{A}_1}.\mathbb{T}^{\mathbf{A}_1}, & & \text{if } \mathbf{l}=\mathbf{A}_1\rightarrow\mathbf{A}_1;\\
\mathbb{T}^{\mathbf{l}} = \mathbf{g} \mid \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\rightarrow\mathbf{l}}\mathbb{T}^{\mathbf{l}_1} \mid \lambda x^{\mathbf{A}_1}\mathbb{T}^{\mathbf{A}_2\rightarrow\mathbf{A}_1}, & & \text{if } \mathbf{l}=\mathbf{A}_1\rightarrow(\mathbf{A}_2\rightarrow\mathbf{A}_1);,\\
\mathbb{T}^{\mathbf{l}} = \quad \mathbb{V}^{\mathbf{l}} \mid \mathbb{T}^{\mathbf{l}_1\rightarrow\mathbf{l}}\mathbb{T}^{\mathbf{l}_1} \mid \lambda x^{\mathbf{l}_2}.\mathbb{T}^{\mathbf{l}_3}, & & \text{if } \mathbf{l}=\mathbf{l}_2\rightarrow\mathbf{l}_3, \mathbf{l}\notin\{\mathbf{A}_1\rightarrow\mathbf{A}_1, \mathbf{A}_1\rightarrow(\mathbf{A}_2\rightarrow\mathbf{A}_1)\}.
\end{array}$$

The well formed formulas of PRED$\lambda\!\rightarrow$ are defined as for PRED$\lambda\!\rightarrow_0$, but with the altered terms.
The postulates of PRED$\lambda\!\rightarrow$ are as for PRED$\lambda\!\rightarrow_0$, plus

$$\Gamma\vdash\varphi,\ \ \varphi=_{\beta\eta}\psi \ \ \Rightarrow\ \ \Gamma\vdash\psi.$$

4.5. Remark. Ordinary predicate logic PRED can be expressed by the pure type system (PTS), see Barendregt[1992],

$\mathcal{S}$	$*^p, *^s, *^f, \Box^p, \Box^s$
$\mathcal{A}$	$*^p : \Box^p, *^s : \Box^s$
$\mathcal{R}$	$(*^p, *^p), (*^s, *^p), (*^s, \Box^p)$
	$(*^s, *^s, *^f), (*^s, *^f, *^f)$

The point is that $*^f$ is a sort for function spaces like $\mathbf{A}_1\rightarrow\mathbf{A}_2$ or $\mathbf{A}_1\rightarrow(\mathbf{A}_2\rightarrow\mathbf{A}_2)$, with $\mathbf{A}_1, \mathbf{A}_2\in *^s$, but not for higher function spaces like $(\mathbf{A}_1\rightarrow\mathbf{A}_2)\rightarrow\mathbf{A}_2$. The system PRED$\lambda\!\rightarrow$ has higher function spaces and moreover it allows quantification over them. PRED$\lambda\!\rightarrow$ can be described more simply as a PTS.

PRED$\lambda\!\rightarrow$

$\mathcal{S}$	$*^p, *^s, \Box^p, \Box^s$
$\mathcal{A}$	$*^p : \Box^p, *^s : \Box^s$
$\mathcal{R}$	$(*^p, *^p), (*^s, *^p), (*^s, \Box^p), (*^s, *^s)$

4.6. Definition. We extend the mappings $[-]^i_r$ and Γ of Definition 4.2 by

$$[\lambda x^{\mathbf{l}_1}.\mathbf{t}^{\mathbf{l}_2}]^i_r = \lambda x^{l_1}.[\mathbf{t}^{\mathbf{l}_2}]^i_r, \quad \Gamma(\lambda x^{\mathbf{l}_1}.\mathbf{t}^{\mathbf{l}_2}) = \Gamma(\mathbf{t}^{\mathbf{l}_2}) - \{l_1 x^{l_1}\}.$$

4.7. Proposition (Soundness of the interpretations for PRED$\lambda\!\rightarrow$). *Let* $\Delta\cup\{\varphi\}\subseteq\mathbb{F}_{\mathrm{PRED}\lambda\rightarrow}$ *, then the following hold for all closed* r.

(i) $\Delta\vdash_{\mathrm{PRED}\lambda\rightarrow}\varphi \ \Rightarrow\ \Gamma^{1,+}_{r,s}, [\Delta]^1_r, \Gamma(\Delta,\varphi)\vdash_{\mathcal{I}\Xi}[\varphi]^1_r$.

(ii) $\Delta\vdash_{\mathrm{PRED}\lambda\rightarrow}\varphi \ \Rightarrow\ \Gamma^{2,+}_{r,s}, [\Delta]^2_r, \Gamma(\Delta,\varphi)\vdash_{\mathcal{I}\mathrm{G}}[\varphi]^2_r M$ *for some* M.

PROOF. We shall not give the proof here. It is similar to the proof of Proposition B2.14 but longer because many more trivial cases have to be considered. □

Now in 4.8-4.15 we shall prove completeness of $\mathcal{I}\mathsf{G}$ for PRED$\lambda\!\rightarrow$ under the indirect translation $[-]^2$. As a corollary we will get in 4.16 completeness of $\mathcal{I}\Xi$ for PRED$\lambda\!\rightarrow$ under the direct translation $[-]^1$.

First we define in 4.8 a grammar $\mathcal{O}_2$, similar to the grammar $\mathcal{O}$ in B3.2. We have $\mathcal{O}_2 \subseteq \mathcal{O}_1$, hence Lemma 3.19 yields that in a proof of $\Gamma \vdash_{\mathcal{I}\mathsf{G}} M$ where $\Gamma \subseteq \overline{\mathcal{O}_2}$ all cut formulas can be eliminated. Then by Lemma 4.14 we get that the proof of completeness of $\mathcal{I}\mathsf{G}$ for PRED$\lambda\!\rightarrow$ is very similar to the proof of completeness of $\mathcal{I}\Xi$ for PRED in B3.11 because all terms occurring in the (cut free) proof are in fact in $\overline{\mathcal{O}_2}$.

4.8. DEFINITION (Grammar $\mathcal{O}_2$ for $\mathcal{I}\mathsf{G}$).

$$\mathcal{L}_2 = A_1 \mid A_2 \mid \mathsf{F}\mathcal{L}_2\mathcal{L}_2;$$

$$\mathcal{V}_2^l = \{x_1^l, x_2^l, \ldots\};$$

$$\begin{array}{lll}
\mathcal{T}_2^l = a \mid \mathcal{V}_2^l \mid \mathcal{T}_2^{\mathsf{F}l_1 l}\mathcal{T}_2^{l_1}, & & \text{if } l \equiv A_1;\\
\mathcal{T}_2^l = \quad \mathcal{V}_2^l \mid \mathcal{T}_2^{\mathsf{F}l_1 l}\mathcal{T}_2^{l_1}, & & \text{if } l \equiv A_2;\\
\mathcal{T}_2^l = f \mid \mathcal{V}_2^l \mid \mathcal{T}_2^{\mathsf{F}l_1 l}\mathcal{T}_2^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_2^{A_1}, & & \text{if } l \equiv \mathsf{F}A_1A_1;\\
\mathcal{T}_2^l = g \mid \mathcal{V}_2^l \mid \mathcal{T}_2^{\mathsf{F}l_1 l}\mathcal{T}_2^{l_1} \mid \lambda x^{A_1}.\mathcal{T}_2^{\mathsf{F}A_2A_1}, & & \text{if } l \equiv \mathsf{F}A_1(\mathsf{F}A_2A_1);\\
\mathcal{T}_2^l = \quad \mathcal{V}_2^l \mid \mathcal{T}_2^{\mathsf{F}l_1 l}\mathcal{T}_2^{l_1} \mid \lambda x^{l_2}.\mathcal{T}_2^{l_3}, & & \text{if } l \equiv \mathsf{F}l_2l_3, l \notin \{\mathsf{F}A_1A_1, \mathsf{F}A_1(\mathsf{F}A_2A_1)\}.
\end{array}$$

$$\begin{array}{l}
\mathcal{P}_2 = P\mathcal{T}_1^{A_1} \mid \Xi l(\lambda x^l.\mathcal{P}_2) \mid \Xi(\mathsf{K}\mathcal{P}_2)(\mathsf{K}\mathcal{P}_2);\\
\mathcal{G}_2 = \mathsf{L}\mathcal{L}_2 \mid \mathsf{H}\mathcal{P}_2 \mid A_i\mathcal{T}_2^{A_i} \mid \Xi l(\lambda x^l.\mathcal{G}_2) \mid \Xi(\mathsf{K}\mathcal{P}_2)(\mathsf{K}\mathcal{G}_2);\\
\mathcal{O}_2 = \mathcal{G}_2 \mid \mathcal{P}_2.
\end{array}$$

Lemmas 3.8 and 3.9 mutatis mutandis also hold for the grammar $\mathcal{O}_2$. The following lemmas 4.9-4.13 concerning the grammar $\mathcal{O}_2$ and connecting it with PRED$\lambda\!\rightarrow$ are similar to Lemmas B3.5-B3.9, with similar easy proofs.

4.9. LEMMA. *Let $\mathcal{W} = \mathcal{T}_2^l, \mathcal{P}_2, \mathcal{G}_2$ or $\mathcal{O}_2, \overline{\mathcal{T}_2^l}, \ldots, \overline{\mathcal{O}_2}$. Then*

$$w{\in}\mathcal{W}, t^l{\in}\mathcal{T}_2^l, x^l{\in}\mathcal{V}_2^l \;\Rightarrow\; w[x^l := t^l]{\in}\mathcal{W}.$$

4.10. LEMMA. *Let $cX_1 \ldots X_n =_{\beta\eta} M$ for some $M{\in}\mathcal{O}_2$ and some constant c. Then $n{\in}\{1,2\}$ and $M \equiv cY_1 \ldots Y_n$ with $Y_i =_{\beta\eta} X_i$.*

4.11. LEMMA. (i) $\Gamma_s^{1,+} \subseteq \overline{\mathcal{G}_2}$.
(ii) $\mathcal{G}_2 \cap \mathcal{P}_2 = \emptyset$.

(iii) $\overline{\mathcal{G}_2} \cap \overline{\mathcal{P}_2} = \emptyset$.

4.12. LEMMA. (i) $[-]^1 : \mathrm{PRED}\lambda\!\!\rightarrow \to \Lambda^s(\mathbf{\Xi}, \mathbf{L})$ *induces bijections*

$$\begin{aligned} [-]^1 &: \mathbb{T}^{\mathbf{l}} \to \mathcal{T}_2^l, \\ [-]^1 &: \mathbb{F}_{\mathrm{PRED}\lambda\rightarrow} \to \mathcal{P}_2. \end{aligned}$$

(ii) *If* $\varphi \in \mathbb{F}_{\mathrm{PRED}\lambda\rightarrow}$ *then* $\Gamma(\varphi) \subseteq \overline{\mathcal{G}_2}$.
(iii) *If* $\Delta \subseteq \mathbb{F}_{\mathrm{PRED}\lambda\rightarrow}$ *and* $\varphi \in \mathbb{F}_{\mathrm{PRED}\lambda\rightarrow}$ *then* $\Gamma_s^{1,+} \cup [\Delta]^1 \cup \Gamma(\Delta, \varphi) \subseteq \overline{\mathcal{O}_2}$.

4.13. LEMMA. *If* $\varphi \in \mathbb{F}_{\mathrm{PRED}\lambda\rightarrow}, x^{\mathbf{l}} \in \mathbb{V}^{\mathbf{l}}, \mathbf{t}^{\mathbf{l}} \in \mathbb{T}^{\mathbf{l}}$ *then*

$$[\varphi]^1[x^l := [\mathbf{t}^{\mathbf{l}}]^1] \equiv [\varphi[x^{\mathbf{l}} := \mathbf{t}^{\mathbf{l}}]]^1.$$

From Lemma 3.19 it follows that in a proof of $\Gamma \vdash_{\mathcal{I}\mathrm{G}} M$ where $\Gamma \subseteq \overline{\mathcal{O}_2}$ also all cut formulas can be eliminated because $\overline{\mathcal{O}_2} \subseteq \overline{\mathcal{O}_1}$. Before we can apply this to prove completeness we only need one more lemma, similar to Lemma 3.22.

4.14. LEMMA. *Let* $\Gamma \subseteq \overline{\mathcal{O}_2}$ *and let* $\Gamma \vdash_{\mathcal{I}\mathrm{G}} N$ *be a cut free proof.Then*
(i) *If the last applied rule in the proof is not the* $\Xi_{\mathrm{i}}-$*rule, nor the* $\mathsf{G}_{\mathsf{L}}-$*rule (modulo the* $\beta\eta-$*rule) then* $N \in \overline{\mathcal{O}_2}$.
(ii) *If* $N = lt$ *with* $l \in \mathcal{L}_2$, $t \in \mathcal{T}_1^l$ *then* $N \in \overline{\mathcal{O}_2}$.

PROOF. The proof is by induction on $\Gamma \vdash N$. The two general cases are easy, so we only need to consider Cases Ξ_{i}, for (ii), and Ξ_{e}.

Case Ξ_{i}. $\Gamma \vdash N = lt = \mathbf{\Xi}XY$ as direct consequence of $\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash Yx$.

Now $l = \mathbf{F}l_1l_2$. $lt = \mathbf{F}l_1l_2t = \mathbf{\Xi}l_1(\lambda x^{l_1}.l_2(tx^{l_1})) = \mathbf{\Xi}XY$. The IH for $\Gamma, l_1x^{l_1} \vdash l_2(tx^{l_1})$ yields $tx^{l_1} \in \overline{\mathcal{T}_2^{l_2}}$. So $t = \lambda x^{l_1}.(tx^{l_1}) \in \overline{\mathcal{T}_2^{\mathbf{F}l_1l_2}}$.

Case Ξ_{e}. $\Gamma \vdash N = YV$ as direct consequence of $\Gamma \vdash \mathbf{\Xi}XY$, $\Gamma \vdash XV$.

By the induction hypothesis for $\Gamma \vdash \mathbf{\Xi}XY$ we have $\mathbf{\Xi}XY \in \overline{\mathcal{O}_2}$. From this follows $YV \in \overline{\mathcal{O}_2}$. (If $\mathbf{\Xi}XY = \mathbf{\Xi}l_1(\lambda x^{l_1}.O)$ we use the IH for $\Gamma \vdash XV$.) □

4.15. PROPOSITION (Completeness of $\mathcal{I}\mathbf{G}$ for PRED$\lambda\!\!\rightarrow$ relative to $[-]^2$).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{G}} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}\lambda\rightarrow} \varphi.$$

PROOF. Again we may suppose that the proof is cut free. By Lemma 4.14 the proof is very similar to the proof of B3.11, because all terms in the proof turn out to be in $\overline{\mathcal{O}_2}$. □

System $\mathcal{I}\mathbf{G}$ is stronger then system $\mathcal{I}\mathbf{\Xi}$. Hence from Proposition 4.15 follows completeness of $\mathcal{I}\mathbf{\Xi}$ for PRED$\lambda\!\!\rightarrow$ under the direct interpretation $[-]^1$.

4.16. Corollary (Completeness of $\mathcal{I}\mathbf{\Xi}$ for PRED$\lambda\!\rightarrow$ relative to $[-]^1$).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}\lambda\rightarrow} \varphi.$$

4.17. Remark (Higher Order Predicate Calculus). Let us define PRED$\lambda\omega$ as PRED$\lambda\!\rightarrow$ but with

$$\mathbb{L}_{\mathrm{PRED}\lambda\omega} = \mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbf{H} \mid \mathbb{L}_{\mathrm{PRED}\lambda\omega}\!\rightarrow\!\mathbb{L}_{\mathrm{PRED}\lambda\omega}.$$

Thus PRED$\lambda\omega$ allows quantification over all functions, but also over all predicates and propositions. As a PTS the system PRED$\lambda\omega$ is PRED$\lambda\!\rightarrow$ plus $(\Box^p, *^p), (\Box^p, \Box^p)$. It is easy to extend the grammar for $\mathcal{I}\mathbf{G}$ to $\mathcal{O}_{\mathrm{PRED}\lambda\omega}$ which has

$$\mathcal{L}_{\mathrm{PRED}\lambda\omega} = A_1 \mid A_2 \mid \mathbf{H} \mid \mathbf{F}\mathcal{L}_{\mathrm{PRED}\lambda\omega}\mathcal{L}_{\mathrm{PRED}\lambda\omega}.$$

Given an extension of $\Gamma^i_{r,s}$ to include $\vdash \mathbf{LH}$, we would expect to prove the completeness of $\mathcal{I}\mathbf{G}$ over PRED$\lambda\omega$, but we have not yet succeeded. In particular we have not been able to prove cut elimination for $\mathbf{LH}\!\in\!\Gamma^i_{r,s}$, so the methods employed here are not all available.

4.18. Remark. In B4.1 we stated that it is possible to work with variants of the systems $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathbf{F}$ and $\mathcal{I}\mathbf{G}$ based on β-conversion only. This still holds. It might seem that in the proof of 3.9 (and the proof of 4.14) we used the $\eta-$axiom, but that is not the case, because by Definition 3.2 not only $ft^{A_1}\!\in\!\mathcal{T}_1^{A_1}$, but also $f\!\in\!\mathcal{T}_1^{\mathbf{F}A_1A_1}$ and similarly $g\!\in\!\mathcal{T}_1^{\mathbf{F}A_1(\mathbf{F}A_2A_1)}$.

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