Systems of Illative Combinatory Logic complete for first-order propositional and predicate calculus[1]

Henk Barendregt*
Martin Bunder†
Wil Dekkers‡

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Abstract

Illative combinatory logic consists of the theory of combinators or lambda calculus extended by extra constants (and corresponding axioms and rules) intended to capture inference. The paper considers systems of illative combinatory logic that are sound for first order propositional and predicate calculus. The interpretation from ordinary logic into the illative systems can be done in two ways: following the propositions-as-types paradigm, in which derivations become combinators, or in a more direct way, in which derivations are not translated. Both translations are closely related in a canonical way. The two direct translations turn out to be complete. The paper fulfills the program of Church [1932,33] and Curry [1930] to base logic on a consistent system of λ-terms or combinators. Hitherto this program had failed because systems of ICL were either too weak (to provide a sound interpretation) or too strong (sometimes even inconsistent).

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*Faculty of Mathematics and Computer Science, Catholic University, Nijmegen, The Netherlands, e-mail: henk@cs.kun.nl.

†Faculty of Informatics, Department of Mathematics, University of Wollongong, NSW Australia, e-mail: Martin_Bunder@info-gw.uow.edu.au.

‡Faculty of Mathematics and Computer Science, Catholic University, Nijmegen, The Netherlands, e-mail: wil@cs.kun.nl.

1. Introduction

The theory of combinators (Curry et al. [1958, 1972]) and the lambda calculus (Church [1941], Barendregt [1984]) are theories that succesfully analyze the notion of effective computability. However, the original founders of these subjects, Curry and Church, also had as aim to provide a basis for logic (and thereby mathematics). Formal systems intended to achieve this are given in Church [1932, 1933] and Curry [1930, 1931, 1932, 1933, 1934a, 1934b, 1935]. Unfortunately, it was shown in Kleene and Rosser [1935] that these systems are inconsistent. In Curry [1942c] the inconsistency of Curry [1934] was simplified. This derivation, now known as 'Curry's paradox', is akin the Russell paradox, but requires no properties of negation. It can be written in only a few lines.

Curry and his school then started a program of defining several systems of illative combinatory logic (ICL) of varying strength, see Curry [1942a]. The goal was to 'find stronger and stronger systems which are consistent and weaker and weaker systems which are inconsistent but strong enough to interpret logic, hoping to end up with a consistent system in which logic can be interpreted' (quotation from Curry and Feys [1958] §8S3, p. 276).

Following this methodology, Bunder [1969], [1973], [1974] introduced restrictions on the rules of the illative constants so that first order propositional and predicate calculus can be interpreted in the resulting systems. Bunder [1983a] also allows much of set theory. In all these systems the usual derivation of Curry's paradox is blocked, but the consistency of these systems remains an open question. That the question is not academic was shown in Bunder [1976] and [1983a], where related illative systems were proved to be inconsistent.

In the rest of this section we give a short introduction to illative combinatory logic by showing the early inconsistent system of Curry [1934]. In section 2 we introduce systems slightly weaker than the ones in Bunder [1973, 1974] but strong enough to interpret logic. We derive roughly the following soundness result

$$\Delta \vdash_{\mathrm{L}} A \;\Rightarrow\; [\Delta] \vdash_{\mathrm{C}} [A],$$

where L represents propositional or predicate logic and $[-]$ one of two possible translations of each system into an ICL system C (there will then be 4 such C's). Of the interpretations one is the propositions-as-types interpretation due to Curry, Howard and de Bruijn, the other is a more direct interpretation.
Finally, in section 2 we show that the two interpretations are canonically related.

In section 3 we derive completeness results for 2 of the 4 systems of ICL. These, again roughly, take the following form

$$[\Delta] \vdash_{\mathrm{C}} [A] \;\Rightarrow\; \Delta \vdash_{\mathrm{L}} A.$$

This completeness result implies the consistency of the ICL's involved.

Illative Combinatory Logic

Now we will present a simple system $\mathcal{I}$ of illative combinatory logic in order to explain the general idea. The system is strong enough to represent the $\{\supset, \forall\}$ fragment of first order intuitionistic predicate calculus.

The intuition behind the system $\mathcal{I}$ is as follows. Terms are type-free lambda terms extended by some extra constants. A term X is considered to have an assertive value. A term XZ can be seen as a statement saying 'Z is of type X' or '$Z{\in}X$' or 'Z satisfies the predicate X'. The term '$\lambda\zeta.X$' corresponds to the class $\{\zeta|X\}$. There is a term $\mathbf{\Xi}$ such that the statement '$\mathbf{\Xi}XY$' is interpreted as '$X \subseteq Y$' or '$(\forall x{\in}X)Yx$'. Using this $\mathbf{\Xi}$ one can define implication and quantification.

1.1. Definition. The system $\mathcal{I}$ is defined as follows.

(i) T, the set of *terms* of $\mathcal{I}$, is given by the following abstract grammar.

$$\mathrm{T} = V \mid \mathbf{\Xi} \mid \mathrm{TT} \mid \lambda V.\mathrm{T}$$

Here V is the syntactical category of variables and $\mathbf{\Xi}$ is a constant. We also write

$$\mathrm{T} = \Lambda(\mathbf{\Xi}),$$

since T is obtained from the set Λ of type-free lambda terms by adding the constant $\mathbf{\Xi}$.

(ii) On T the usual notion of $\beta\eta$-reduction is given by the contraction rules

$$\begin{aligned} (\lambda x.M)N &\rightarrow M[x := N], \\ \lambda x.Mx &\rightarrow M \text{ if } x{\notin}\mathrm{FV}(M). \end{aligned}$$

Here $\mathrm{FV}(M)$ is the set of free variables of M. The resulting (more step) $\beta\eta$-*reduction* and $\beta\eta$-*convertibility* relation are denoted by $\twoheadrightarrow$ and $=$. Syntactic equality is denoted by $\equiv$.

(iii) A *statement* of $\mathcal{I}$ is just an element of T. A *basis* is a set of statements.

(iv) Let Γ be a basis and X be a statement, then X is *derivable* from Γ, notation $\Gamma \vdash X$, if $\Gamma \vdash X$ can be produced by the following natural deduction system.

$$\boxed{\begin{aligned} X{\in}\Gamma &\Rightarrow \Gamma \vdash X, \\ \Gamma \vdash X,\ X = Y &\Rightarrow \Gamma \vdash Y, \\ \Gamma \vdash \mathbf{\Xi}XY,\ \Gamma \vdash XZ &\Rightarrow \Gamma \vdash YZ, \\ \Gamma, Xx \vdash Yx,\ x{\notin}\mathrm{FV}(\Gamma, X, Y) &\Rightarrow \Gamma \vdash \mathbf{\Xi}XY. \end{aligned}}$$

In the last rule x is some variable. The system is based on $\beta\eta$-conversion. Therefore this last rule could be replaced by

$$\Gamma, X \vdash Y \; x \notin \mathrm{FV}(\Gamma) \;\Rightarrow\; \Gamma \vdash \mathbf{\Xi}(\lambda x.X)(\lambda x.Y).$$

1.2. Definition. For $X, Y \in \mathrm{T}$ write
(i) $X \supset Y \equiv \mathbf{\Xi}(\mathbf{K}X)(\mathbf{K}Y)$,
(ii) $\forall u \in X.Y \equiv \mathbf{\Xi}X(\lambda u.Y)$.

1.3. Proposition. *The following holds for the system $\mathcal{I}$.*
(i) $\Gamma \vdash X \supset Y,\ \Gamma \vdash X \;\Rightarrow\; \Gamma \vdash Y$.
(ii) $\Gamma, X \vdash Y \;\Rightarrow\; \Gamma \vdash X \supset Y$.
(iii) $\Gamma \vdash \forall u \in X.Y,\ \Gamma \vdash Xt \;\Rightarrow\; \Gamma \vdash Y[u := t]$.
(iv) $\Gamma, Xu \vdash Y,\ u \notin \mathrm{FV}(\Gamma, X) \;\Rightarrow\; \Gamma \vdash \forall u \in X.Y$.

Now it is possible to interpret the $\{\supset, \forall\}$ fragment of first order intuitionistic predicate logic into $\mathcal{I}$. For example a sentence like

$$\forall x(Rx \supset Rx)$$

holding in a universe A, is translated as the statement

$$(\forall x \in A.Rx \supset Rx)$$

which is $\mathbf{\Xi}A(\lambda x.\mathbf{\Xi}(\mathbf{K}(Rx))(\mathbf{K}(Rx)))$ and is provable in $\mathcal{I}$.

Unfortunately the interpretation is not complete (i.e. if the translation of a formula φ is provable in $\mathcal{I}$, then φ itself is provable in logic) because the system $\mathcal{I}$ is not consistent: every statement X (i.e. every term) can be derived in $\mathcal{I}$ (from the empty basis).

1.4. Proposition (Curry's paradox). *Let X be a statement of $\mathcal{I}$. Then $\vdash X$.*

Proof. Let X be given. Take

$$Y \equiv (\lambda y.(yy) \supset X)(\lambda y.(yy) \supset X).$$

Then $Y = Y \supset X$. Therefore the following derivation shows that $\vdash X$.

$$\begin{array}{rcll} Y & \vdash & Y, & \\ Y & \vdash & Y \supset X, & \text{since } Y = Y \supset X, \\ Y & \vdash & X, & \text{by 1.3(i)}, \\ & \vdash & Y \supset X, & \text{by 1.3(ii)}, \\ & \vdash & Y, & \text{since } Y \supset X = Y, \\ & \vdash & X, & \text{by 1.3(i)}. \quad \square \end{array}$$

Note that the derivation of X is related to the proof of the theorem of Löb [1955].

In the next section the illative system $\mathcal{I}$ will be formulated more carefully so that the system becomes consistent and in fact complete over ordinary logic.

2. Sound interpretations of logics in ICL's.

In the introduction we stated that logic can be interpreted in two ways in ICL's. In fact this can be done both for the propositional and predicate calculus, so there will be four related illative systems. The interpretation will be done for the $\{\supset\}$ (respectively $\{\supset, \forall\}$) fragment of intuitionistic logic. This is the most essential part of logic and the direct interpretation ($[-]$ below) can be extended to include the logical operators $\neg$, $\&$, $\vee$ and $\exists$. In second order logic these operators are definable from $\supset, \forall$, so both our interpretations can be extended into sound (and probably complete) interpretations of second order logical calculi.

Now we display the two logical calculi that will be interpreted.

2.1. Definition. Let PROP be the $\supset$ fragment of intuitionistic propositional logic determined as follows.

(i) The set of formulas of PROP, notation $\mathbb{F}_{\mathrm{PROP}}$, is defined by the following abstract syntax

$$\mathbb{F}_{\mathrm{PROP}} = \mathbb{V} \mid \mathbb{F}_{\mathrm{PROP}} \supset \mathbb{F}_{\mathrm{PROP}}.$$

Here $\mathbb{V}$ is a set of propositional variables.

(ii) Let $\Gamma \subseteq \mathbb{F}_{\mathrm{PROP}}$ and $\varphi \in \mathbb{F}_{\mathrm{PROP}}$. Then $\Gamma \vdash_{\mathrm{PROP}} \varphi$ is defined by the following system of natural deduction.

PROP

$$\begin{array}{rcl} \varphi \in \Gamma & \Rightarrow & \Gamma \vdash \varphi; \\ \Gamma \vdash \varphi \supset \psi,\ \ \Gamma \vdash \varphi & \Rightarrow & \Gamma \vdash \psi; \\ \Gamma, \varphi \vdash \psi & \Rightarrow & \Gamma \vdash \varphi \supset \psi. \end{array}$$

2.2. Definition. Let PRED be the $\{\supset, \forall\}$ fragment of first order many-sorted intuitionistic predicate calculus of a given signature s.

Below as an example, we will treat a version of PRED with s the signature of the structure

$$< \mathbf{A}_1, \mathbf{A}_2, \mathbf{f}, \mathbf{g}, \mathbf{P}, \mathbf{a} >$$

with

$$\begin{array}{rl} \mathbf{A}_1, \mathbf{A}_2 & \text{nonempty sets;} \\ \mathbf{f} : \mathbf{A}_1 \to \mathbf{A}_1 & \text{a unary function;} \\ \mathbf{g} : \mathbf{A}_1 \to \mathbf{A}_2 \to \mathbf{A}_1 & \text{a binary function;} \\ \mathbf{P} \subseteq \mathbf{A}_1 & \text{a unary relation;} \\ \mathbf{a} \in \mathbf{A}_1 & \text{a constant.} \end{array}$$

(All results also hold for arbitrary signatures.)

(i) The set of terms of PRED, notation $\mathbb{T}_{\text{PRED}}$, is defined by the following abstract syntax.

$$\begin{array}{rcl}\mathbb{T}_{\text{PRED}} & = & \mathbb{T}_{\mathbf{A}_1} \mid \mathbb{T}_{\mathbf{A}_2} \\ \mathbb{T}_{\mathbf{A}_1} & = & \mathbb{V}^{\mathbf{A}_1} \mid \mathbf{a} \mid \mathbf{f}\mathbb{T}_{\mathbf{A}_1} \mid \mathbf{g}\mathbb{T}_{\mathbf{A}_1}\mathbb{T}_{\mathbf{A}_2} \\ \mathbb{T}_{\mathbf{A}_2} & = & \mathbb{V}^{\mathbf{A}_2}.\end{array}$$

(ii) The set of formulas of PRED, notation $\mathbb{F}_{\text{PRED}}$, is defined by the following abstract syntax.

$$\mathbb{F}_{\text{PRED}} = \mathbf{P}\mathbb{T}_{\mathbf{A}_1} \mid \mathbb{F}_{\text{PRED}} \supset \mathbb{F}_{\text{PRED}} \mid \forall \mathbb{V}^{\mathbf{A}_i}\mathbb{F}_{\text{PRED}}$$

(iii) $\Gamma \vdash_{\text{PRED}} \varphi$ is axiomatised by the following system of natural deduction.

PRED

$$\begin{array}{rcl} \varphi \in \Gamma & \Rightarrow & \Gamma \vdash \varphi; \\ \Gamma \vdash \varphi \supset \psi, \;\; \Gamma \vdash \varphi & \Rightarrow & \Gamma \vdash \psi; \\ \Gamma, \varphi \vdash \psi & \Rightarrow & \Gamma \vdash \varphi \supset \psi; \\ \Gamma \vdash \forall x^{\mathbf{A}_i}\varphi, \;\; t \in \mathbb{T}_{\mathbf{A}_i} & \Rightarrow & \Gamma \vdash \varphi[x^{\mathbf{A}_i} := t]; \\ \Gamma \vdash \varphi, \;\; x^{\mathbf{A}_i} \notin \text{FV}(\Gamma) & \Rightarrow & \Gamma \vdash \forall x^{\mathbf{A}_i}\varphi. \end{array}$$

Now the systems PROP and PRED will be interpreted in ICL's. In order to block the proof of the Curry paradox, Bunder [1969, 1973, 1974] modified the system $\mathcal{I}$ by restricting the $\mathbf{\Xi}$-introduction rule and adding some other axioms and a rule. The resulting system $\mathcal{I}_0$ was strong enough to provide sound interpretations of PROP and PRED while the proof of the Curry paradox was blocked. However the problems of the completeness of the interpretation and even of the consistency of $\mathcal{I}_0$ remained open. (The system $\mathcal{I}_0$ will be described later.)

We will give modified versions of $\mathcal{I}_0$ in which the logics can be embedded in a sound way by two kinds of embeddings. The first kind is 'direct' and the second kind is according to the 'propositions-as-types' and 'proofs-as-terms' paradigm, see Barendregt [1992], §5.1, §5.4. As there are two logical systems, PROP and PRED, there will be four systems of ICL. These systems are called $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathbf{F}$, and $\mathcal{I}\mathbf{G}$ respectively. Their use for the two kinds of interpretation is as follows. Let $[\ ]^1$ be the direct, and $[\ ]^2$ the propositions-as-types translation. Then the following table shows the systems of ICL that are used for the two translations of PROP and PRED.

	$[\ ]^1$	$[\ ]^2$
PROP	$\mathcal{I}\mathsf{P}$	$\mathcal{I}\mathsf{F}$
PRED	$\mathcal{I}\mathsf{\Xi}$	$\mathcal{I}\mathsf{G}$

For example

$$[\ ]^2 : \mathrm{PRED} \to \mathcal{I}\mathsf{G}.$$

The four systems ICL will be described now and moreover their relative strengths will be compared.

2.3. Definition. Let $\mathrm{T} = \Lambda(\mathsf{\Xi}, \mathsf{L})$, be the set of type-free lambda terms extended by the extra constants $\mathsf{\Xi}$ and L.

(i) Define the following terms in T.

$$\begin{array}{rcl} \mathsf{P} & \equiv & \lambda xy.\mathsf{\Xi}(\mathsf{K}x)(\mathsf{K}y), \\ \mathsf{F} & \equiv & \lambda xyz.\mathsf{\Xi}x(y{\circ}z), \\ \mathsf{G} & \equiv & \lambda xyz.\mathsf{\Xi}x(\mathsf{S}yz), \\ \mathsf{H} & \equiv & \mathsf{L}{\circ}\mathsf{K}, \end{array}$$

where $\mathsf{K} \equiv \lambda pq.p$, $M{\circ}N \equiv \lambda x.M(Nx)$ and $\mathsf{S} \equiv \lambda pqr.pr(qr)$.
Write $X \supset Y$ for $\mathsf{P}XY$.

(ii) Define the following four systems of illative combinatory logic $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{\Xi}$, $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$. All four systems have as rules the following:

All systems

$$\begin{array}{rcl} X{\in}\Gamma & \Rightarrow & \Gamma \vdash X; \\ \Gamma \vdash X, X =_{\beta\eta} Y & \Rightarrow & \Gamma \vdash Y. \end{array}$$

The four systems have the following specific rules:

$\mathcal{I}\mathsf{P}$

$$\begin{array}{llcl} \mathsf{P_e} & \Gamma \vdash X \supset Y,\ \ \Gamma \vdash X & \Rightarrow & \Gamma \vdash Y; \\ \mathsf{P_i} & \Gamma, X \vdash Y,\ \ \Gamma \vdash \mathsf{H}X & \Rightarrow & \Gamma \vdash X \supset Y; \\ \mathsf{P_H} & \Gamma, X \vdash \mathsf{H}Y,\ \ \Gamma \vdash \mathsf{H}X & \Rightarrow & \Gamma \vdash \mathsf{H}(X \supset Y). \end{array}$$

$\mathcal{I}\mathbf{\Xi}$

Ξ_{e}	$\Gamma \vdash \mathbf{\Xi} XY,\ \ \Gamma \vdash XV$	$\Rightarrow$	$\Gamma \vdash YV;$
Ξ_{i}	$\Gamma, Xx \vdash Yx,\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y)$	$\Rightarrow$	$\Gamma \vdash \mathbf{\Xi} XY;$
Ξ_{H}	$\Gamma, Xx \vdash \mathbf{H}(Yx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y)$	$\Rightarrow$	$\Gamma \vdash \mathbf{H}(\mathbf{\Xi} XY).$

$\mathcal{I}\mathbf{F}$

F_{e}	$\Gamma \vdash \mathbf{F} XYZ,\ \ \Gamma \vdash XV$	$\Rightarrow$	$\Gamma \vdash Y(ZV);$
F_{i}	$\Gamma, Xx \vdash Y(Zx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y, Z)$	$\Rightarrow$	$\Gamma \vdash \mathbf{F} XYZ;$
F_{L}	$\Gamma, Xx \vdash \mathbf{L}Y,\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y)$	$\Rightarrow$	$\Gamma \vdash \mathbf{L}(\mathbf{F} XY).$

$\mathcal{I}\mathbf{G}$

G_{e}	$\Gamma \vdash \mathbf{G} XYZ,\ \ \Gamma \vdash XV$	$\Rightarrow$	$\Gamma \vdash YV(ZV);$
G_{i}	$\Gamma, Xx \vdash Yx(Zx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y, Z)$	$\Rightarrow$	$\Gamma \vdash \mathbf{G} XYZ;$
G_{L}	$\Gamma, Xx \vdash \mathbf{L}(Yx),\ \ \Gamma \vdash \mathbf{L}X,\ \ x \notin \mathrm{FV}(\Gamma, X, Y)$	$\Rightarrow$	$\Gamma \vdash \mathbf{L}(\mathbf{G} XY).$

To get a taste for what will follow, we give some examples of interpretations of tautologies in the ICL's.

2.4. Examples. (i) The formula $p \supset p$ of PROP is translated as $p \supset p$ in $\mathcal{I}\mathbf{P}$. The fact that $p \supset p$ is indeed a wff of PROP is expressed in $\mathcal{I}\mathbf{P}$ as $\mathbf{H}p \vdash \mathbf{H}(p \supset p)$, which should be interpreted as 'if p is a proposition, then so is $p \supset p$.' So $\mathbf{H}$ functions as the class of propositions. It was used in Curry [1942a], Bunder [1969] and others, to block the derivation of Curry's paradox. Aczel [1980] uses $\mathbf{H}$ as in Bunder [1969] and in $\mathcal{I}\mathbf{P}$.

The fact that $p \supset p$ is derivable in PROP is interpreted in $\mathcal{I}\mathbf{P}$ as $\mathbf{H}p \vdash p \supset p$. This should be interpreted as 'if p is a proposition, then $p \supset p$ is derivable'.

(ii) The same formula $p \supset p$ is translated in $\mathcal{I}\mathbf{F}$ as $\mathbf{F}pp$. The fact that $p \supset p$ is a wff of PROP is expressed in $\mathcal{I}\mathbf{F}$ by $\mathbf{L}p \vdash \mathbf{L}(\mathbf{F}pp)$, which should be interpreted as 'if p is a type, then $\mathbf{F}pp$ is a type'. The type $\mathbf{F}pp$ is intuitively the function space type $p \to p$. The fact that $p \supset p$ is derivable in PROP is interpreted in $\mathcal{I}\mathbf{F}$ by making the type $\mathsf{F}pp$ 'inhabited' by the expression $\lambda y.y$ (formulas-as-types and terms-as-derivations interpretation).

$$\mathsf{L}p \vdash \mathsf{F}pp(\lambda y.y)$$

(iii) Similarly, consider the formula $\forall x^{\mathsf{A}}(\mathbf{P}x \supset \mathbf{P}x)$ of PRED. Interpreted in $\mathcal{I}\mathsf{\Xi}$ this becomes

$$\mathsf{L}A, \mathsf{F}A\mathsf{H}P \vdash \mathsf{H}(\mathsf{\Xi}A(\lambda x.Px \supset Px)).$$

The intuitive meaning of $\mathsf{F}A\mathsf{H}P$ is 'P is of type $A{\to}\mathsf{H}$', that is: P is a map from A into the propositions, hence a predicate on A. (In generalised type systems the basis $\mathsf{L}A, \mathsf{F}A\mathsf{H}P$ would be written as the context $A{:}{*}^s, P{:}A{\to}{*}^p$, see Barendregt [1992], especially the systems λPRED and λP.) The fact that $\forall x^{\mathsf{A}}(\mathbf{P}x \supset \mathbf{P}x)$ is derivable in PRED becomes

$$\mathsf{L}A, \mathsf{F}A\mathsf{H}P \vdash \mathsf{\Xi}A(\lambda x.Px \supset Px)$$

which is derivable in $\mathcal{I}\mathsf{\Xi}$.

(iv) The formula $\forall x^{\mathsf{A}}(\mathbf{P}x \supset \mathbf{P}x)$ in PRED translated in $\mathcal{I}\mathsf{G}$ becomes

$$\mathsf{L}A, \mathsf{F}A\mathsf{L}P \vdash \mathsf{L}(\mathsf{G}A(\lambda x.Px \supset Px)),$$

which should be interpreted as 'if A is a type and P is in $A{\to}\mathsf{L}$, then $\mathsf{G}A(\lambda x.Px \supset Px)$ is a type'. In the PTS language of λP this is

$$A{:}{*}, P{:}A{\to}{*} \vdash (\Pi x{:}A.Px{\to}Px) : {*}.$$

The fact that $\forall x^{\mathsf{A}}(\mathbf{P}x \supset \mathbf{P}x)$ is a tautology is interpreted in $\mathcal{I}\mathsf{G}$ by the inhabitation of the type $\mathsf{G}A(\lambda x.Px \supset Px)$.

$$\mathsf{L}A, \mathsf{F}A\mathsf{L}P \vdash \mathsf{G}A(\lambda x.Px \supset Px)(\lambda x.\lambda y.y).$$

2.5. Notes. (i) $\mathcal{I}_0$ is essentially $\mathcal{I}\mathsf{\Xi}$ plus the following:

$$\vdash \mathsf{\Xi IH}, \ \vdash \mathsf{L}A_1, \ \vdash \mathsf{L}A_2 \ \text{ and } \ \vdash \mathsf{LH}.$$

By the axiom '$\vdash \mathsf{LH}$' one can interpret second order propositional and predicate logic. For example by rule Ξ_{i} one gets

$$\vdash \mathsf{\Xi H}(\lambda p.p{\to}p) \quad (\equiv \forall p{\in}\mathsf{H}.p{\to}p).$$

So one can quantify over propositions. By '$\vdash \mathsf{\Xi IH}$' one can derive $p \vdash \mathsf{H}p$ and even $p \vdash \mathsf{H}(\mathsf{H}p)$. Here one has a mixture of statements in the language and in the metalanguage. This gives problems in the study of completeness and maybe it is better to skip this axiom '$\vdash \mathsf{\Xi IH}$', as we have done in the systems $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{\Xi}$, $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$.

(ii) In the ICL's not only tautologies can be derived but also so called syntactical conditions. For example in signature s one has $\mathbf{f}(\mathbf{a}){\in}\mathbb{T}_{\mathbf{A}_1}$ and $\mathbf{P}(\mathbf{f}(\mathbf{a})){\in}\mathbb{F}_{\mathrm{PRED}}$. These translate as $\mathsf{F}A_1A_1f, A_1a \vdash A_1(fa)$ and $\mathsf{F}A_1A_1f, A_1a, \mathsf{F}A_1\mathsf{H}P \vdash \mathsf{H}(P(fa))$ respectively.

The following lemma is useful for determining the relative strength of the four systems.

2.6. LEMMA. *For all $X, Y{\in}\mathrm{T}$ one has the following in $\Lambda(\mathbf{\Xi}, \mathsf{L})$.*

(i) $\mathsf{F}(\mathsf{K}X)(\mathsf{K}Y) = \mathsf{K}(\mathsf{P}XY)$.

(ii) $\mathsf{G}X(\mathsf{K}{\circ}Y) = \mathsf{K}(\mathbf{\Xi}XY)$.

(iii) $\mathsf{F}XY = \mathsf{G}X(\mathsf{K}Y)$.

PROOF. (i) $\mathsf{F}(\mathsf{K}X)(\mathsf{K}Y) = \lambda z.\mathbf{\Xi}(\mathsf{K}X)((\mathsf{K}Y){\circ}z) = \lambda z.\mathbf{\Xi}(\mathsf{K}X)(\mathsf{K}Y) = \mathsf{K}(\mathsf{P}XY)$, since $(\mathsf{K}Y){\circ}z = \lambda x.\mathsf{K}Y(zx) = \lambda x.Y = \mathsf{K}Y$.

(ii) $\mathsf{G}X(\mathsf{K}{\circ}Y) = \lambda z.\mathbf{\Xi}X(\mathsf{S}(\mathsf{K}{\circ}Y)z) = \lambda z.\mathbf{\Xi}XY = \mathsf{K}(\mathbf{\Xi}XY)$, since $\mathsf{S}(\mathsf{K}{\circ}Y)z = \lambda c.(\mathsf{K}{\circ}Y)c(zc) = \lambda c.Yc = Y$.

(iii) $\mathsf{F}XY = \lambda z.\mathbf{\Xi}X(Y{\circ}z) = \lambda z.\mathbf{\Xi}X(\mathsf{S}(\mathsf{K}Y)z) = \mathsf{G}X(\mathsf{K}Y)$ since $\mathsf{S}(\mathsf{K}Y)z = \lambda v.\mathsf{K}Yv(zv) = \lambda v.Y(zv) = Y{\circ}z$. □

2.7. PROPOSITION. *The systems $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$ are related as follows.*

$$\begin{tikzcd} \mathcal{I}\mathsf{F} \arrow[r] & \mathcal{I}\mathsf{G} \\ \mathcal{I}\mathsf{P} \arrow[u] \arrow[r] & \mathcal{I}\mathbf{\Xi} \arrow[u] \end{tikzcd}$$

where $\longrightarrow$ denotes nondecreasing strength, i.e. $s_1 \longrightarrow s_2$ means that for all Γ, X

$$\Gamma \vdash_{s_1} X \;\Rightarrow\; \Gamma \vdash_{s_2} X.$$

PROOF. We will show that if $s_1 \longrightarrow s_2$ in the diagram then every rule of s_1 can be derived in s_2.

(i) Case $\mathcal{I}\mathsf{P} \to \mathcal{I}\mathbf{\Xi}$. The rules P_e, P_i and P_H follow from respectively Ξ_e, Ξ_i and Ξ_H by the substitutions of $\mathsf{K}X$ for X and $\mathsf{K}Y$ for Y.

(ii) Case $\mathcal{I}\mathsf{P} \to \mathcal{I}\mathsf{F}$. Then P_e, P_i and P_H follow from F_e, F_i and F_L by the substitutions of $\mathsf{K}X$ for X and $\mathsf{K}Y$ for Y and Lemma 2.6(i).

(iii) Case $\mathcal{I}\mathsf{F} \to \mathcal{I}\mathsf{G}$. Now F_e, F_i and F_L follow from G_e, G_i and G_L by the substitution of $\mathsf{K}Y$ for Y and Lemma 2.6(iii).

(iv) Case $\mathcal{I}\mathbf{\Xi} \to \mathcal{I}\mathsf{G}$. Then Ξ_e, Ξ_i and Ξ_H follow from G_e, G_i and G_L by the substitution of $\mathsf{K}{\circ}Y$ for Y and Lemma 2.6(ii). □

Now we will show formally how the logics PROP and PRED can be interpreted in the illative systems. We start with PROP.

2.8. DEFINITION. Let r be a closed term in $\Lambda(\mathbf{\Xi},\mathbf{L})$. Two maps (for $i=1,2$)

$$[-]^i_r : \mathbb{F}_{\mathrm{PROP}} \to \Lambda(\mathbf{\Xi},\mathbf{L})$$

and two maps

$$\Gamma^i_r : \mathbb{F}_{\mathrm{PROP}} \to \text{illative contexts}$$

are defined by the following table. (Note that these illative contexts are effectively grammatical conditions on the variables (propositional, individual) that appear in a proposition.)

φ	$[\varphi]^1_r$	$\Gamma^1_r(\varphi)$	$[\varphi]^2_r$	$\Gamma^2_r(\varphi)$
p	rp	$\mathbf{H}(rp)$	rp	$\mathbf{L}(rp)$
$\psi\supset\chi$	$[\psi]^1_r\supset[\chi]^1_r$	$\Gamma^1_r(\psi),\Gamma^1_r(\chi)$	$\mathbf{F}[\psi]^2_r[\chi]^2_r$	$\Gamma^2_r(\psi),\Gamma^2_r(\chi)$

The r in the above and in 2.12 can be replaced by $\mathbf{I}$ (i.e. omitted). However in 2.15 we use it to derive a relation between the two interpretations.

2.9. LEMMA. *Let $\varphi\in\mathbb{F}_{\mathrm{PROP}}$ and let $\{p_1,\dots,p_n\}$ be the set of (free) propositional variables in φ. Then*
(i) $\Gamma^1_r(\varphi)=\{\mathbf{H}(rp_1),\dots,\mathbf{H}(rp_n)\}$.
(ii) $\Gamma^2_r(\varphi)=\{\mathbf{L}(rp_1),\dots,\mathbf{L}(rp_n)\}$.
(iii) $\mathbf{H}\varphi,\mathbf{H}\psi\vdash_{\mathcal{I}\mathrm{P}}\mathbf{H}(\varphi\supset\psi)$.
(iv) $\mathbf{L}\varphi,\mathbf{L}\psi\vdash_{\mathcal{I}\mathrm{F}}\mathbf{L}(\mathbf{F}\varphi\psi)$.
(v) $\Gamma^1_r(\varphi)\vdash_{\mathcal{I}\mathrm{P}}\mathbf{H}[\varphi]^1_r$.
(vi) $\Gamma^2_r(\varphi)\vdash_{\mathcal{I}\mathrm{F}}\mathbf{L}[\varphi]^2_r$.

PROOF. (i), (ii) By induction on the length of φ.
(iii), (iv) By $\mathsf{P_H}$ and $\mathsf{F_L}$.
(v) By (i) and (iii).
(vi) By (ii) and (iv). □

2.10. DEFINITION. Let $\Delta\subseteq\mathbb{F}_{\mathrm{PROP}}$.
(i) $[\Delta]^1_r=\{[\varphi]^1_r\mid\varphi\in\Delta\}$.
(ii) $[\Delta]^2_r=\{[\varphi]^2_r x_\varphi\mid\varphi\in\Delta\}$, with x_φ a fresh variable chosen uniquely for φ.
(iii) $\Gamma^i_r(\Delta)=\{\Gamma^i_r(\varphi)\mid\varphi\in\Delta\}$.
(iv) $\Gamma^i_r(\Delta,\varphi)=\Gamma^i_r(\Delta),\Gamma^i_r(\varphi)$.

If Δ is a set of assumptions in a deduction in PROP or PRED, then $[\Delta]^i_r$ is the set of translated assumptions. Note that $[\varphi]^2_r$ in a sense represents a class. Each $[\varphi]^2_r x_\varphi$ then represents the condition that $[\varphi]^2_r$ is inhabited, corresponding to the fact that φ is assumed to be true.
The $\Gamma^i_r(\Delta)$ are grammatical conditions required for the variables of Δ.

In the proof of the following proposition there is an unexpected difficulty in showing the soundness of modus ponens. The difficulty can be avoided by a trick, which however does not work for PRED as we will see and explain.

2.11. Proposition (Soundness of the interpretations for PROP). *Let* $\Delta\cup\{\varphi\} \subseteq \mathbb{F}_{\mathrm{PROP}}$. *Then one has the following for all closed* r.

(i) $\Delta \vdash_{\mathrm{PROP}} \varphi \;\Rightarrow\; [\Delta]^1_r, \Gamma^1_r(\Delta,\varphi) \vdash_{\mathcal{I}\mathrm{P}} [\varphi]^1_r$.

(ii) $\Delta \vdash_{\mathrm{PROP}} \varphi \;\Rightarrow\; \exists M{\in}\Lambda[\;[\Delta]^2_r, \Gamma^2_r(\Delta,\varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi]^2_r M\;\;]$.

Proof. (i) By induction on the derivations in PROP.

If $\varphi{\in}\Delta$, then the result holds by the first rule for $\vdash$ in $\mathcal{I}\mathbf{P}$.

If $\Delta \vdash \varphi$ is a direct consequence of $\Delta \vdash \psi \supset \varphi$ and $\Delta \vdash \psi$ then the induction hypothesis (IH) implies (leaving out the super and subscripts):

$$[\Delta], \Gamma(\Delta, \psi \supset \varphi) \vdash [\psi] \supset [\varphi],$$

$$[\Delta], \Gamma(\Delta, \psi) \vdash [\psi].$$

Therefore, by rule P_{e} one has $[\Delta], \Gamma(\Delta,\varphi), \Gamma(\psi) \vdash [\varphi]$.
If $\{q_1,\dots,q_m\}$ is the set of propositional variables occurring in ψ but not in φ or Δ and p is a propositional variable occurring in φ, then we have

$$[\Delta], \Gamma(\Delta,\varphi), \mathbf{H}(rq_1),\dots,\mathbf{H}(rq_m) \vdash [\varphi],$$

where $\mathbf{H}(rp){\in}\Gamma(\Delta,\varphi)$. Substituting p for each of $q_1,\dots,q_m$ gives

$$[\Delta], \Gamma(\Delta,\varphi) \vdash [\varphi].$$

If $\Delta \vdash \varphi$ is $\Delta \vdash \psi \supset \chi$ and is a direct consequence of $\Delta, \psi \vdash \chi$, then by the IH one has

$$[\Delta], [\psi], \Gamma(\Delta, \psi, \chi) \vdash [\chi].$$

By Lemma 2.9(v)

$$\Gamma(\psi) \vdash \mathbf{H}[\psi].$$

Hence we have $[\Delta], \Gamma(\Delta), \Gamma(\psi), \Gamma(\chi) \vdash [\psi \supset \chi]$. So by the definition of Γ

$$[\Delta], \Gamma(\Delta), \Gamma(\psi \supset \chi) \vdash [\psi \supset \chi].$$

(ii) As for (i) except that for every $p \in \Delta \cup \{\varphi\}$ in the derivation $[p]^i_r$ will have attached a variable x_p and every compound proposition a compound term. For example if $\varphi \in \Delta$ then $[\Delta]^2_r \vdash [\varphi]^2_r x_\varphi$ and in the modus ponens case if

$$[\Delta], \Gamma(\Delta, \psi \supset \varphi) \vdash_{\mathcal{I}\mathrm{F}} (\mathsf{F}[\psi][\varphi])M$$

and

$$[\Delta], \Gamma(\Delta, \psi) \vdash_{\mathcal{I}\mathrm{F}} [\psi]N$$

then

$$[\Delta], \Gamma(\Delta, \psi, \psi \supset \varphi) \vdash_{\mathcal{I}\mathrm{F}} [\varphi](MN). \quad \square$$

2.12. Definition. (i) $\Lambda^s(\mathbf{\Xi}, \mathsf{L})$ is $\Lambda(\mathbf{\Xi}, \mathsf{L})$ extended by the extra constants A_1, A_2, P, f, g, a associated with the signature s of the manysorted structure of our example. Because we are going to interprete many sorted predicate logic with sorts $\mathbf{A}_1, \mathbf{A}_2$, it is useful to have among the free variables of the λ-calculus infinite sets $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_i = \{x_i, y_i, z_i \ldots\}$. x, y, z denote arbitrary variables.

(ii) Let r be a closed term in $\Lambda^s(\mathbf{\Xi}, \mathsf{L})$. Two maps (for $i = 1, 2$)

$$[-]^i_r : \mathbb{F}_{\mathrm{PRED}} \to \Lambda^s(\mathbf{\Xi}, \mathsf{L})$$

and a map

$$\Gamma : \mathbb{F}_{\mathrm{PRED}} \to \text{illative contexts}$$

are defined by the following tables:

t	$[t]^i_r$	$\Gamma(t)$
$x^{\mathbf{A}_j}$	x_j	$A_j x_j$
$\mathbf{a}$	a	$\emptyset$
$\mathbf{f}s$	$f[s]^i_r$	$\Gamma(s)$
$\mathbf{g}st$	$g[s]^i_r[t]^i_r$	$\Gamma(s), \Gamma(t)$

φ	$[\varphi]^1_r$	$[\varphi]^2_r$	$\Gamma(\varphi)$
$\mathbf{P}t$	$r(P[t]^1_r)$	$r(P[t]^2_r)$	$\Gamma(t)$
$\psi \supset \chi$	$[\psi]^1_r \supset [\chi]^1_r$	$\mathsf{F}[\psi]^2_r[\chi]^2_r$	$\Gamma(\psi), \Gamma(\chi)$
$\forall x^{\mathbf{A}_i}\psi$	$\mathbf{\Xi} A_i(\lambda x_i.[\psi]^1_r)$	$\mathsf{G} A_i(\lambda x_i.[\psi]^2_r)$	$\Gamma(\psi) - \{A_i x_i\}$

(iii)

$$\Gamma^1_{r,s} =< \mathsf{L}A_1, \mathsf{L}A_2, \mathsf{F}A_1A_1f, \mathsf{F}A_1(\mathsf{F}A_2A_1)g, \mathsf{F}A_1\mathsf{H}(r\circ P), A_1a >$$

,

$$\Gamma^2_{r,s} =< \mathsf{L}A_1, \mathsf{L}A_2, \mathsf{F}A_1A_1f, \mathsf{F}A_1(\mathsf{F}A_2A_1)g, \mathsf{F}A_1\mathsf{L}(r\circ P), A_1a >$$

and

$$\Gamma^{i,+}_{r,s} = \Gamma^i_{r,s} \cup \{A_2x_2\} \text{ where } x_2{\in}\mathcal{V}_2 \text{ is some variable.}$$

The definitions $\Gamma^i_{r,s}$ and $\Gamma^{i,+}_{r,s}$ of course refer to our example of a many-sorted predicate calculus with signature s.
It is essential to add A_2x_2 to $\Gamma^i_{r,s}$ (and if required similarly for other sorts), to avoid the problem of possibly empty domains. It would be natural that

$$\Delta \vdash_{\text{PRED}} \varphi \quad \Rightarrow \quad \Gamma^1_{r,s}, [\Delta]^1_r, \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_r.$$

However this is not true. The similar problem for PTS's (see Barendregt [1992]) was first noted by E. Barendsen [1989]. The point is that in ordinary (minimal, intuitionistic or classical) logic it is always assumed that the universes $\mathbf{A}_1, \mathbf{A}_2 \dots$ of the structure are supposed to be non-empty. For example

$$(\forall x^{\mathbf{A}}(Px{\to}Q)){\to}(\forall x^{\mathbf{A}}Px){\to}Q$$

is provable in PRED, but only valid in structures with $\mathbf{A} \neq \emptyset$. In so-called *free logic* one also allows structures with empty domains. This logic has been axiomatised by Peremans [1949] and Mostowski [1951]. What is unexpected is that the problem turns up in the case of modus ponens. (c.f. the proof of Proposition 2.14)

2.13. Lemma. *Let* $\varphi{\in}\mathbb{F}_{\text{PRED}}$.
(i) *If* $t{\in}\mathbb{T}_{\mathbf{A}_j}$ *then in* $\mathcal{I}\boldsymbol{\Xi}$ *or* $\mathcal{I}\mathsf{G}$, *one has* $\Gamma(t), \Gamma^i_{r,s} \vdash A_j[t]^i_r$.
(ii) $\Gamma^1_{r,s}, \Gamma(\varphi) \vdash_{\mathcal{I}\Xi} \mathsf{H}[\varphi]^1_r$.
(iii) $\Gamma^2_{r,s}, \Gamma(\varphi) \vdash_{\mathcal{I}\text{G}} \mathsf{L}[\varphi]^2_r$.

Proof. (i) By induction on the length of t, using the statements $A_1a, \mathsf{F}A_1A_1f$ and $\mathsf{F}A_1(\mathsf{F}A_2A_1)g$ in $\Gamma^i_{r,s}$.
(ii) If $\varphi = \mathbf{P}t$ where $t{\in}\mathbb{T}_{\mathbf{A}_1}$, then by $\Gamma^1_{r,s} \vdash \mathsf{F}A_1\mathsf{H}(r\circ P)$, (i) and $\Gamma(\varphi) = \Gamma(t)$ we have $\Gamma^1_{r,s}, \Gamma(\varphi) \vdash_{\mathcal{I}\Xi} \mathsf{H}(r(P[t]))$ as required. The remaining cases are as in Lemma 2.9(v).
(iii) As in (ii) and Lemma 2.9(vi). $\square$

2.14. Proposition (Soundness of the interpretations for PRED). *Let* $\Delta \cup \{\varphi\} \subseteq \mathbb{F}_{\text{PRED}}$*, then the following hold for all closed* r*.*

(i) $\Delta \vdash_{\text{PRED}} \varphi \quad \Rightarrow \quad \Gamma^{1;+}_{r,s}, [\Delta]^1_r, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_r$.

(ii) $\Delta \vdash_{\text{PRED}} \varphi \quad \Rightarrow \quad \Gamma^{2;+}_{r,s}, [\Delta]^2_r, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\text{G}} [\varphi]^2_r M$ *for some* M*.*

Proof. (i) The induction on the proof of $\Delta \vdash_{\text{PRED}} \varphi$, is as in the proof of Proposition 2.11(i). When, in the case of modus ponens, terms $A_1 x_1$ for variables $x_1 \in \text{FV}(\psi) - \text{FV}(\varphi)$ need to be removed from the left of the $\vdash$ we replace x_1 by a. If terms $A_2 x_i$ for $x_i \in \text{FV}(\psi) - \text{FV}(\varphi)$, we replace x_i by x_2 and note that $A_2 x_2 \in \Gamma^{1;+}_{r,s}$.

(ii) The induction on the proof of $\Delta \vdash_{\text{PRED}} \varphi$, is as in the proof of Proposition 2.11(ii) with $\mathsf{G}[\psi](\lambda x.[\varphi])$ instead of $\mathsf{F}[\psi][\varphi]$. Variables may need to be replaced as in (i).

2.15. Proposition (The relation between the two interpretations).

(i) *For* $\varphi \in \mathbb{F}_{\text{PROP}}$ *one has* $\mathsf{K}[\varphi]^1_r = [\varphi]^2_{\mathsf{K} \circ r}$.

(ii) *For* $\varphi \in \mathbb{F}_{\text{PRED}}$ *one has* $\mathsf{K}[\varphi]^1_r = [\varphi]^2_{\mathsf{K} \circ r}$.

Proof. (i) By induction on the length of φ.

$$\mathsf{K}[p]^1_r = \mathsf{K}(rp) = [p]^2_{\mathsf{K}\circ r},$$

$$\mathsf{K}[\varphi \supset \psi]^1_r = \mathsf{K}([\varphi]^1_r \supset [\psi]^1_r) = \mathsf{F}(\mathsf{K}[\varphi]^1_r)(\mathsf{K}[\psi]^1_r),$$

by Lemma 2.6(i), so by the IH

$$\mathsf{K}[\varphi \supset \psi]^1_r = \mathsf{F}([\varphi]^2_{\mathsf{K}\circ r})([\psi]^2_{\mathsf{K}\circ r}) = [\varphi \supset \psi]^2_{\mathsf{K}\circ r}.$$

(ii) By induction on the length of φ as in (i) but also using Lemma 2.6(ii). □

3. Completeness of two of the interpretations

In this section we derive completeness for the interpretations $[\ ]^1 : \text{PROP} \to \mathcal{I}\mathbf{P}$ and $[\ ]^1 : \text{PRED} \to \mathcal{I}\boldsymbol{\Xi}$. We conjecture completeness for the interpretations $[\ ]^2 : \text{PROP} \to \mathcal{I}\mathbf{F}$ and $[\ ]^2 : \text{PRED} \to \mathcal{I}\mathbf{G}$, but we have not been able to prove it.[1]

We start with the proof of the completeness for $\mathcal{I}\boldsymbol{\Xi}$ relative to PRED. This occupies subsections 3.1-3.11. The proof for $\mathcal{I}\mathbf{P}$ relative to PROP in 3.12-3.14 goes in a similar way but is much easier.

[1]After the paper had been sent to the journal we succeeded to prove completeness for $[\ ]^2 : \text{PROP} \to \mathcal{I}\mathbf{F}$, but completeness for $[\ ]^2 : \text{PRED} \to \mathcal{I}\mathbf{G}$ is still open.

Completeness for $\mathcal{I}\boldsymbol{\Xi}$ relative to PRED

We will show

$$\forall r[\ \Gamma^{1,+}_{r,s}, [\Delta]^1_r, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_r\ ] \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

Here the signature s and the context $\Gamma^{1,+}_{r,s}$ are as in 2.2 and 2.12; again the result can easily be generalised to other signatures.

It is sufficient to show

$$\Gamma^{1,+}_{r,s}, [\Delta]^1_r, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1_r \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi$$

for a special r. We choose $r \equiv \mathsf{I}$, i.e. we omit r and we prove

$$\Gamma^{1,+}_{s}, [\Delta]^1, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PRED}} \varphi.$$

where the definitions of $\Gamma^{1,+}_s$ and $[-]^1$ are obtained from 2.12 by everywhere omitting r.

The proof goes in two steps. First we define a *grammar* in order to analyze the terms M such that $\Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} M$. Then the completeness is shown by means of this analysis. Instead of $\vdash_{\mathcal{I}\Xi}$ we shall mostly write $\vdash$.

3.1. Remark. That it is not obvious that completeness holds is because not only translations of tautologies can be derived in the ICL's, but also syntactical statements. Let $\Gamma = \Gamma^{1,+}_s, [\Delta]^1, \Gamma(\Delta,\varphi)$. Then we can derive in $\mathcal{I}\boldsymbol{\Xi}$ sequents of the form

$$\Gamma \vdash [\varphi]$$

where $[\varphi]$ is (the translation of) a logical formula, but also

$$\Gamma \vdash \mathsf{H}(Pt)$$

where $\mathsf{H}(Pt)$ corresponds to the syntactical statement $\mathbf{Pt}{\in}\mathbb{F}_{\mathrm{PRED}}$ in the metalanguage. In

$$\Gamma \vdash \mathsf{L}A_i$$

$\mathsf{L}A_i$ corresponds to the fact that A_i is one of the sets in the signature s. Even a mixture is possible:

$$\mathsf{H}p \vdash p \supset \mathsf{H}p.$$

Using the grammar it will be shown that such mixed statements do not interfere with the logic. The translations of logical formulas will form a class $\overline{\mathcal{P}}$ (propositions) in our grammar, and the other statements a class $\overline{\mathcal{G}}$ (grammatical conditions).

3.2. DEFINITION (Grammar for derivable statements in $\mathcal{I}\mathbf{\Xi}$).

$$
\begin{array}{rcl}
\mathcal{T} & = & \mathcal{T}_1 \mid \mathcal{T}_2 \\
\mathcal{T}_1 & = & \mathcal{V}_1 \mid a \mid f\mathcal{T}_1 \mid g\mathcal{T}_1\mathcal{T}_2 \\
\mathcal{T}_2 & = & \mathcal{V}_2
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{P} & = & P\mathcal{T}_1 \mid \mathbf{\Xi}A_i(\lambda x_i.\mathcal{P}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P}) \\
\mathcal{G} & = & \mathbf{L}A_i \mid A_i\mathcal{T}_i \mid \mathbf{\Xi}A_i(\lambda x_i.\mathcal{G}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}) \mid \mathbf{L}(\mathbf{K}\mathcal{P}) \\
\mathcal{O} & = & \mathcal{G} \mid \mathcal{P}
\end{array}
$$

$$
\begin{array}{rcl}
\overline{\mathcal{T}_i} & = & \{M \mid \exists N{\in}\mathcal{T}_i \mid N =_{\beta\eta} M\} \\
\overline{\mathcal{P}} & = & \{M \mid \exists N{\in}\mathcal{P} \mid N =_{\beta\eta} M\} \\
\overline{\mathcal{G}} & = & \{M \mid \exists N{\in}\mathcal{G} \mid N =_{\beta\eta} M\} \\
\overline{\mathcal{O}} & = & \overline{\mathcal{P}} \cup \overline{\mathcal{G}}.
\end{array}
$$

3.3. REMARKS. (i) All elements of $\mathcal{O}$ are in $\beta\eta$-normal form if we read $\mathbf{K}M$ as $\lambda y.M$. So all elements of $\overline{\mathcal{O}}$ have a (unique) $\beta\eta$-normal form.

(ii) $\overline{\mathcal{P}}$ and $\overline{\mathcal{G}}$ do not exhaust the possible theorems of $\mathcal{I}\mathbf{\Xi}$, e.g.

$$
\begin{array}{rcl}
\mathbf{H}(\mathbf{H}p) & \vdash & \mathbf{H}(\mathbf{H}p \supset p) \\
\mathbf{H}p, \mathbf{H}(\mathbf{H}p) & \vdash & \mathbf{H}(p \supset \mathbf{H}p).
\end{array}
$$

3.4. NOTATION. *Let $M{\in}\overline{\mathcal{O}}$ with normal form N and let u be a variable. Then we write $u{\notin_{\beta\eta}}\mathrm{FV}(M)$ for $u{\notin}\mathrm{FV}(N)$.*

Now in 3.5-3.10 we state some technical results that are needed in the completeness proof. The main proposition is 3.10, stating that only terms in $\overline{\mathcal{O}}$ can be derived from $\Gamma_s^{1,+}$; this gives the required analysis.

3.5. LEMMA. (i) *Let $\mathcal{W} = \mathcal{T}_1, \mathcal{T}_2, \mathcal{P}, \mathcal{G}$, or $\mathcal{O}$. Then*

$$w{\in}\mathcal{W}, t_i{\in}\mathcal{T}_i, x_i{\in}\mathcal{V}_i \;\Rightarrow\; w[x_1 := t_1, x_2 := t_2]{\in}\mathcal{W}.$$

(ii) *Let $\overline{\mathcal{W}} = \overline{\mathcal{T}_1}, \overline{\mathcal{T}_2}, \overline{\mathcal{P}}, \overline{\mathcal{G}}$, or $\overline{\mathcal{O}}$. Then*

$$w{\in}\overline{\mathcal{W}}, t_i{\in}\overline{\mathcal{T}_i}, x_i{\in}\mathcal{V}_i \;\Rightarrow\; w[x_1 := t_1, x_2 := t_2]{\in}\overline{\mathcal{W}}.$$

PROOF. (i) By a simple induction.

(ii) From (i) and

$$M_1 =_{\beta\eta} M_2, N_1 =_{\beta\eta} N_2 \;\Rightarrow\; M_1[x := N_1] =_{\beta\eta} M_2[x := N_2]. \qquad \square$$

3.6. LEMMA. *Let $cX_1 \ldots X_n =_{\beta\eta} M$ for some $M{\in}\mathcal{O}$ and some constant c. Then $n{\in}\{1, 2\}$ and $M \equiv cY_1 \ldots Y_n$ with $Y_i =_{\beta\eta} X_i$.*

PROOF. By Church-Rosser and the fact that all elements of $\mathcal{O}$ are in $\beta\eta$-normal form.

3.7. LEMMA. (i) $\Gamma_s^{1,+} \subset \overline{\mathcal{G}}$.
(ii) $\mathcal{G} \cap \mathcal{P} = \emptyset$.
(iii) $\overline{\mathcal{G}} \cap \overline{\mathcal{P}} = \emptyset$.

PROOF. (i) $\mathbf{F}A_1A_1f = \mathbf{\Xi}A_1(\lambda x_1.A_1(fx_1)) \in\mathcal{G}$ because $A_1(fx_1)\in\mathcal{G}$.
Moreover $\mathbf{F}A_1(\mathbf{F}A_2A_1)g = \mathbf{\Xi}A_1(\lambda x_1.\mathbf{\Xi}A_2(\lambda x_2.A_1(gx_1x_2))) \in\mathcal{G}$.
Finally $\mathbf{F}A_1\mathbf{H}P = \mathbf{\Xi}A_1(\lambda x_1.\mathbf{H}(Px_1)) \in\mathcal{G}$ because $\mathbf{H}(Px_1)\in\mathcal{G}$.
(ii) By an easy induction.
(iii) From (ii) by Church-Rosser and the fact that the elements of $\mathcal{O} = \mathcal{G} \cup \mathcal{P}$ are in $\beta\eta$-normal form. $\square$

3.8. LEMMA. (i) $[-]^1 : \mathrm{PRED} \to \Lambda^s(\mathbf{\Xi}, \mathbf{L})$ *induces bijections*

$$\begin{aligned} [-]^1 &: \mathbb{T}_{\mathbf{A}_1} \to \mathcal{T}_1, \\ [-]^1 &: \mathbb{T}_{\mathbf{A}_2} \to \mathcal{T}_2, \\ [-]^1 &: \mathbb{F}_{\mathrm{PRED}} \to \mathcal{P}. \end{aligned}$$

(ii) *If* $\varphi\in\mathbb{F}_{\mathrm{PRED}}$ *then* $[\varphi]^1\in\mathcal{P}$ *and* $\Gamma(\varphi) \subset \mathcal{G}$.
(iii) *If* $\Delta \subset \mathbb{F}_{\mathrm{PRED}}$ *and* $\varphi\in\mathbb{F}_{\mathrm{PRED}}$ *then* $\Gamma_s^{1,+} \cup [\Delta]^1 \cup \Gamma(\Delta, \varphi) \subset \overline{\mathcal{O}}$.

PROOF. (i) and (ii) by easy inductions.
(iii) from (ii) and 3.7(i). $\square$

3.9. LEMMA. *If* $\varphi\in\mathbb{F}_{\mathrm{PRED}}, x^{\mathbf{A}_i}\in\mathbb{V}^{\mathbf{A}_i}, t_{\mathbf{A}_i}\in\mathbb{T}_{\mathbf{A}_i}, x_i\in\mathcal{V}_i$ *then*

$$[\varphi]^1[x_i := [t_{\mathbf{A}_i}]^1] \equiv [\varphi[x^{\mathbf{A}_i} := t_{\mathbf{A}_i}]]^1$$

PROOF. By an easy induction. $\square$

3.10. PROPOSITION.

$$\Gamma \vdash M, \Gamma \subset \overline{\mathcal{O}} \;\Rightarrow\; M\in\overline{\mathcal{O}}.$$

PROOF. We use induction loading and show

$$\Gamma \vdash M,\ \Gamma \subset \overline{\mathcal{O}} \;\&\; u\notin_{\beta\eta}\mathrm{FV}(\Gamma) \;\Rightarrow\; M\in\overline{\mathcal{O}} \;\&\; u\notin_{\beta\eta}\mathrm{FV}(M) \quad (*)$$

by induction on the derivation of $\Gamma \vdash M$. We only consider the three $\mathbf{\Xi}$ - rules; the other two rules are easy.

Case Ξ_{e}. $\Gamma \vdash M$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \mathbf{\Xi}XY$, $\Gamma \vdash XV$.
By the IH one has $\mathbf{\Xi}XY\in\overline{\mathcal{O}}, u\notin_{\beta\eta}\mathrm{FV}(\mathbf{\Xi}XY)$ & $XV\in\overline{\mathcal{O}}, u\notin_{\beta\eta}\mathrm{FV}(XV)$. We distinguish two cases according to the form of $\mathbf{\Xi}XY$, using Lemma 3.6.

Subcase Ξ_e(a). $X=A_i, Y=\lambda x_i.O$, with $O\in\mathcal{O}$. Now $A_iV\in\overline{\mathcal{O}}$ & $u\notin_{\beta\eta}\mathrm{FV}(A_iV)$. So $V=t_i$ where $t_i\in\mathcal{T}_i$ and $u\notin\mathrm{FV}(t_i)$. Hence by Lemma 3.5(i)

$$M=(\lambda x_i.O)t_i=O[x_i:=t_i]\in\mathcal{O}$$

and

$$u\notin_{\beta\eta}\mathrm{FV}(O[x_i:=t_i]),$$

because $u\notin\mathrm{FV}(\lambda x_i.O)\cup\mathrm{FV}(t_i)$.

Subcase Ξ_e(b). $X=\mathbf{K}p, Y=\mathbf{K}O$, with $p\in\mathcal{P}, O\in\mathcal{O}$. Now $YV=O$ where $u\notin\mathrm{FV}(O)$.

Case Ξ_i. $\Gamma\vdash M$ is $\Gamma\vdash\mathbf{\Xi}XY$ as direct consequence of
$\Gamma\vdash\mathbf{L}X,\ \ \Gamma,Xx\vdash Yx$ with $x\notin\mathrm{FV}(\Gamma,X,Y)$.

By the IH one has $\mathbf{L}X\in\overline{\mathcal{O}}$ and $u\notin_{\beta\eta}\mathrm{FV}(\mathbf{L}X)$. We distinguish two cases according to the form of X.

Subcase Ξ_i(a). $X=A_i$. Now x is any variable so we may assume that $x\in\mathcal{V}_i, u\not\equiv x$. Then $A_ix\in\overline{\mathcal{O}}$ and $u\notin\mathrm{FV}(A_ix)$, hence by the IH one has

$$Yx\in\overline{\mathcal{O}}\text{ and }u\notin_{\beta\eta}\mathrm{FV}(Yx).$$

Let $Yx=O\in\mathcal{O}$. Then $M=\mathbf{\Xi}A_i(\lambda x.O)$ where $u\notin_{\beta\eta}\mathrm{FV}(\lambda x.O)$.

Subcase Ξ_i(b). $X=\mathbf{K}p$. Then $\Gamma,p\vdash Yx$. One has $x\notin\mathrm{FV}(\Gamma,p)$, $u\notin_{\beta\eta}\mathrm{FV}(\Gamma,p)$ because $u\notin_{\beta\eta}\mathrm{FV}(\mathbf{L}X)$. So by the IH one has

$$Yx=O\in\mathcal{O},\text{ where }x\notin\mathrm{FV}(O),u\notin\mathrm{FV}(O).$$

Hence $Y=\mathbf{K}O$ and $M=\mathbf{\Xi}(\mathbf{K}p)(\mathbf{K}O)$ where $u\notin\mathrm{FV}(pO)$.

Case $\Xi_\mathbf{H}$. $\Gamma\vdash M$ is $\Gamma\vdash\mathbf{H}(\mathbf{\Xi}XY)$ as direct consequence of
$\Gamma\vdash\mathbf{L}X,\ \ \Gamma,Xx\vdash\mathbf{H}(Yx)$ with $x\notin\mathrm{FV}(\Gamma,X,Y)$.

The proof is similar to the proof for case Ξ_i. We now get $\mathbf{H}(Yx)\in\overline{\mathcal{O}}$ hence $Yx\in\overline{\mathcal{P}}$ and

$$M=\mathbf{H}(\mathbf{\Xi}A_i(\lambda x_i.p))\text{ with }u\notin\mathrm{FV}(\lambda x_i.p)\text{ in case }\Xi_\mathbf{H}\text{(a)},$$
$$M=\mathbf{H}(\mathbf{\Xi}(\mathbf{K}p_1)(\mathbf{K}p_2))\text{ with }u\notin\mathrm{FV}(p_1p_2)\text{ in case }\Xi_\mathbf{H}\text{(b)}.\quad\square$$

3.11. Proposition (Completeness for $\mathcal{I}\boldsymbol{\Xi}$ relative to PRED).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta,\varphi) \vdash_{\mathcal{I}\Xi} [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}} \varphi.$$

Proof. $\Gamma_s^{1,+} \subset \overline{\mathcal{G}}$ by Lemma 3.7(i) and $\Gamma(\Delta,\varphi) \subset \overline{\mathcal{G}}$ by Lemma 3.8(ii). Hence it is sufficient to prove

$$\Gamma_g, [\Delta]^1 \vdash_{\mathcal{I}\Xi} M,\;\; \Gamma_g \subset \overline{\mathcal{G}},\;\; M = [\varphi]^1 \;\Rightarrow\; \Delta \vdash_{\mathrm{PRED}} \varphi. \qquad (**)$$

Write $\Gamma = \Gamma_g, [\Delta]^1$. Then $\Gamma \subset \overline{\mathcal{O}}$, hence $M \in \overline{\mathcal{O}}$ by Proposition 3.10.

The proof of $(**)$ goes by induction.

Case 1. $\Gamma \vdash M$ because $M \in \Gamma$.

$M = [\varphi]^1 \in \mathcal{P}$ by Lemma 3.8(i) so, as $\mathcal{P} \subset \overline{\mathcal{P}}$ and $\overline{\mathcal{P}} \cap \overline{\mathcal{G}} = \emptyset$, $M \in [\Delta]^1$. As the elements of $[\Delta]^1$ are in nf as are those of $\mathcal{P}$ one has $[\varphi]^1 \in [\Delta]^1$, and by Lemma 3.8(i) $\varphi \in \Delta$. Hence $\Delta \vdash_{\mathrm{PRED}} \varphi$.

Case 2. $\Gamma \vdash M$ is a direct consequence of $\Gamma \vdash N$ and $M = N$.

Now $N = M = [\varphi]^1$ and by the IH for N one has $\Delta \vdash_{\mathrm{PRED}} \varphi$.

Case Ξ_{e}. $\Gamma \vdash M$ is $\Gamma \vdash YV$ as direct consequence of $\Gamma \vdash \boldsymbol{\Xi} XY$, $\Gamma \vdash XV$.

As $\boldsymbol{\Xi} XY \in \mathcal{O}$ by Proposition 3.10 we need consider only 4 cases.

Subcase Ξ_{e}(a). $X = A_i, Y = \lambda x_i.p$. Now $\Gamma \vdash A_i V \in \overline{\mathcal{O}}$ by Proposition 3.10. Therefore $V = t_i = [t_{\mathbf{A}_i}]^1$. Since $p \in \mathcal{P}$ we can write $p = [\psi]^1$ by 3.8(i). Therefore $[\varphi]^1 = YV = (\lambda x_i.p)[t_{\mathbf{A}_i}]^1 = [\psi]^1[x_i := [t_{\mathbf{A}_i}]^1] = [\psi[x^{\mathbf{A}_i} := t_{\mathbf{A}_i}]]^1$. So $[\varphi]^1 \equiv [\psi[x^{\mathbf{A}_i} := t_{\mathbf{A}_i}]]^1$ Hence $\varphi \equiv \psi[x^{\mathbf{A}_i} := t_{\mathbf{A}_i}]$. $\boldsymbol{\Xi} XY = \boldsymbol{\Xi} A_i(\lambda x_i.p) = [\forall x^{\mathbf{A}_i}\psi]^1$. By the IH one has $\Delta \vdash_{\mathrm{PRED}} \forall x^{\mathbf{A}_i}\psi$. So $\Delta \vdash_{\mathrm{PRED}} \psi[x^{\mathbf{A}_i} := t_{\mathbf{A}_i}] \equiv \varphi$.

Subcase Ξ_{e}(b). $X = \mathsf{K}p_1, Y = \mathsf{K}p_2$. Then $[\varphi]^1 = M = YV = p_2 \in \mathcal{P}$ and by 3.8(i) we can write $p_1 = [\varphi_1]^1$. Therefore $\boldsymbol{\Xi} XY = \boldsymbol{\Xi}(\mathsf{K}p_1)(\mathsf{K}p_2) = [\varphi_1 \supset \varphi]^1$. By the IH one has

$$\Delta \vdash_{\mathrm{PRED}} \varphi_1 \supset \varphi.$$

Also $XV = p_1 = [\varphi_1]^1$, so by the IH one has

$$\Delta \vdash_{\mathrm{PRED}} \varphi_1.$$

Therefore it follows by modus ponens that

$$\Delta \vdash_{\mathrm{PRED}} \varphi.$$

Subcase Ξ_{e}(c). $X = A_i, Y = \lambda x_i.g$. Since $\Gamma \vdash A_i V \in \overline{\mathcal{O}}$ one has $V \in \overline{\mathcal{T}_i}$. Therefore $M = YV = g[x_i := V] \in \overline{\mathcal{G}}$. So $M \neq [\varphi]^1$ for all φ by Lemma 3.7(iii). Contradiction.

Subcase Ξ_e(d). $X = \mathsf{K}p, Y = \mathsf{K}g$. Now $M = YV = g$, so $M \neq [\varphi]^1$ for all φ. Contradiction.

Case Ξ_i. $\Gamma \vdash M$ is $\Gamma \vdash \mathbf{\Xi}XY$ as direct consequence of
$\Gamma \vdash \mathsf{L}X$, $\Gamma, Xx \vdash Yx$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

As $\mathbf{\Xi}XY = [\varphi]^1 \in \mathcal{P}$ we need consider only 2 cases.

Subcase Ξ_i(a). $X = A_i, Y = \lambda x_i.p$. Let $p = [\psi]^1$. Then $M = [\forall x^{\mathbf{A}_i}\psi]^1$. Now x is any variable, so we may assume $x \in \mathcal{V}_i$. Then $Xx = \mathbf{A}_i x \in \overline{\mathcal{G}}$.
As $\Gamma, Xx \vdash Yx$ one has $\Gamma, Xx \vdash [\psi]^1$. So $\Delta \vdash_{\mathrm{PRED}} \psi$ by the IH. Now x does not occur in Γ, so

$$\Delta \vdash_{\mathrm{PRED}} \forall x^{\mathbf{A}_i}\psi.$$

Subcase Ξ_i(b). $X = \mathsf{K}p_1, Y = \mathsf{K}p_2$. Let $p_1 = [\varphi_1]^1, p_2 = [\varphi_2]^1$. Then $M = [\varphi_1 \supset \varphi_2]^1$. Now $\Gamma, Xx \vdash Yx$ is $\Gamma, p_1 \vdash p_2$. So by the IH one has $\Delta, \varphi_1 \vdash_{\mathrm{PRED}} \varphi_2$. Hence

$$\Delta \vdash_{\mathrm{PRED}} \varphi_1 \supset \varphi_2.$$

Case Ξ_{H}. $\Gamma \vdash M$ is $\Gamma \vdash \mathsf{H}(\mathbf{\Xi}XY)$ as direct consequence of
$\Gamma \vdash \mathsf{L}X$, $\Gamma, Xx \vdash \mathsf{H}(Yx)$ with $x \notin \mathrm{FV}(\Gamma, X, Y)$.

This case is not applicable because $M \notin \overline{\mathcal{P}}$. $\square$

Completeness for $\mathcal{I}\mathsf{P}$ relative to PROP

The proof of this completeness follows the same pattern as the proof of the completeness for $\mathcal{I}\mathbf{\Xi}$ relative to PRED, but it is easier. As in that proof it is sufficient to take $r \equiv \mathsf{I}$, i.e. we omit r.

3.12. Definition (Grammar for derivable statements for $\mathcal{I}\mathsf{P}$).

$$\begin{aligned}
\mathcal{P} &= \mathcal{V} \mid \mathcal{P} \supset \mathcal{P} \\
\mathcal{G} &= \mathsf{H}\mathcal{P} \mid \mathcal{P} \supset \mathcal{G} \\
\mathcal{O} &= \mathcal{G} \mid \mathcal{P}.
\end{aligned}$$

$\overline{\mathcal{P}}, \overline{\mathcal{G}}$ and $\overline{\mathcal{O}}$ are then defined as in 3.2.

3.13. Proposition.

$$\Gamma \vdash_{\mathcal{I}\mathrm{P}} M,\ \Gamma \subset \overline{\mathcal{O}} \ \Rightarrow\ M \in \overline{\mathcal{O}}.$$

Proof. By induction on the derivation of $\Gamma \vdash_{\mathcal{I}\mathrm{P}} M$. The various cases correspond to the two initial cases and cases Ξ_e(b), Ξ_i(b) and Ξ_{H}(b) of the proof of Proposition 3.10. $\square$

3.14. Proposition (Completeness for $\mathcal{I}\mathsf{P}$ relative to PROP).

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{P}} [\varphi]^1 \ \Rightarrow\ \Delta \vdash_{\mathrm{PROP}} \varphi.$$

PROOF. This consists of the initial cases and cases $\Xi_e(b)$, $\Xi_e(d)$ $\Xi_i(b)$ of the proof of Proposition 3.11 and the Ξ_{H} case with $\mathbf{K}p_1$ for X and $\mathbf{K}p_2$ for Y. As before $[-]^1$ is 1-1 and $\mathcal{G}\cap\mathcal{P}=\emptyset$ and if $X\in\mathcal{O}$ then $\Gamma^1(X)\subset\mathcal{G}$. $\square$

4. Remarks and open problems

4.1. REMARKS. (i) The systems $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{\Xi}$, $\mathcal{I}\mathbf{F}$ and $\mathcal{I}\mathbf{G}$ are based on $\beta\eta$-conversion. It is possible to work with variants of these systems based on β-conversion only. Change the rules for $\mathcal{I}\mathbf{\Xi}$ as follows.

$\mathcal{I}\mathbf{\Xi}$

	$X\in\Gamma$	$\Rightarrow$	$\Gamma\vdash X;$
	$\Gamma\vdash X, X=_\beta Y$	$\Rightarrow$	$\Gamma\vdash Y;$
Ξ_e	$\Gamma\vdash\mathbf{\Xi}(\lambda x.X)(\lambda x.Y),\ \ \Gamma\vdash(\lambda x.X)V$	$\Rightarrow$	$\Gamma\vdash(\lambda x.Y)V;$
Ξ_i	$\Gamma, X\vdash Y,\ \ \Gamma\vdash\mathbf{L}(\lambda x.X),\ \ x\notin\mathrm{FV}(\Gamma)$	$\Rightarrow$	$\Gamma\vdash\mathbf{\Xi}(\lambda x.X)(\lambda x.Y);$
Ξ_{H}	$\Gamma, X\vdash\mathbf{H}Y,\ \ \Gamma\vdash\mathbf{L}(\lambda x.X),\ \ x\notin\mathrm{FV}(\Gamma)$	$\Rightarrow$	$\Gamma\vdash\mathbf{H}(\mathbf{\Xi}(\lambda x.X)(\lambda x.Y)).$

and similarly for $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{F}$ and $\mathcal{I}\mathbf{G}$. Then in the proof of the completeness for $\mathcal{I}\mathbf{\Xi}$ relative to PRED only minor changes need to be made, like replacing A_i, X, Xx and $\beta\eta$ by $\lambda x_i.A_ix_i$, $\lambda x.X$, X and β respectively.
Similar changes should be made in the proof of the completeness for $\mathcal{I}\mathbf{P}$ relative to PROP. The reader is invited to verify all details.

(ii) The additional primitive $\mathbf{L}$ added to $\Lambda(\mathbf{\Xi})$, was not strictly necessary. We could have used the definition $\mathbf{L}=\mathbf{W\Xi}$ and simplified our grammar for derivable statements in $\mathcal{I}\mathbf{\Xi}$ in 3.2 in the following way:

$$\begin{array}{rcl}\mathcal{P} & = & P\mathcal{T}_1 \mid \mathbf{\Xi}A_i(\lambda x_i\mathcal{P}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P})\\ \mathcal{G} & = & A_i\mathcal{T}_i \mid \mathbf{\Xi}A_i(\lambda x_i.\mathcal{G}) \mid \mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}).\end{array}$$

Note that now $\mathbf{L}(\mathbf{K}\mathcal{P})=\mathbf{\Xi}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P})$, so $\mathbf{L}(\mathbf{K}\mathcal{P})$ shifts from $\mathcal{G}$ to $\mathcal{P}$! Similarly for $\mathcal{I}\mathbf{P}$ $\mathbf{H}$ can be defined as $\mathbf{WP}$ as was done in Curry [1942a].

(iii) In the work of Seldin and others $\mathbf{L}$ is defined as $\mathbf{FEH}$ where $\mathbf{E}$ is a universal class. Under this definition $\mathbf{H}p$ and $\mathbf{L}(\mathbf{K}p)$ are interderivable, but our proof of Proposition 3.10 fails.

(iv) The title of the paper refers to combinatory logic, but the systems used are based on lambda calculus throughout. The illative systems could have been based on combinatory logic using an appropriate bracket abstraction algorithm.

(v) For historical and other remarks concerning the combinators $\mathsf{\Xi}, \mathsf{P}, \mathsf{F}$ and G, see Hindley and Seldin [1986] : Ch.17 for $\mathsf{\Xi}$ and P, Ch.13 and Ch.15 for F, and Ch 16 §§C, D for G.

4.2. Open problems. The following is a list of open problems.

(i)

$$\begin{array}{ll}\text{Is the interpretation} \quad [\ \ ]^2 & : \mathrm{PROP} \to \mathcal{I}\mathsf{F} \text{ complete?} \\ \text{Is the interpretation} \quad [\ \ ]^2 & : \mathrm{PRED} \to \mathcal{I}\mathsf{G} \text{ complete?}\end{array}$$

(ii) Is $\mathcal{I}\mathsf{F}$ a conservative extension of $\mathcal{I}\mathsf{P}$ and $\mathcal{I}\mathsf{G}$ a conservative extension of $\mathcal{I}\mathsf{\Xi}$?

(iii) Adding as axiom LH to $\mathcal{I}\mathsf{\Xi}$ one can interpret second order propositional and predicate logic. Is this interpretation complete?

(iv) Is the extension $\mathcal{I}_0$ of $\mathcal{I}\mathsf{\Xi}$ in 2.5 complete? Are similar extensions of $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$ complete?

4.3. Remark. The system $\mathcal{I}_0$ is consistent. This can be seen in the following way. Let

$$\begin{array}{lll}\mathcal{G} & = & \mathcal{V} \mid \mathsf{L}A_i \mid \mathsf{LH} \mid \mathsf{\Xi IH} \mid \mathsf{\Xi}A_i(\lambda y.\mathcal{G}) \mid \mathsf{\Xi H}(\lambda y.\mathcal{G}) \mid \mathsf{\Xi}(\mathsf{K}\mathcal{G})(\mathsf{K}\mathcal{G}) \mid A_i\mathcal{G} \mid \mathsf{H}\mathcal{G}. \\ \overline{\mathcal{G}} & = & \{M \mid \exists N \in \mathcal{G} \mid N =_{\beta\eta} M\}.\end{array}$$

Then one can prove

$$\Gamma \vdash M, \Gamma \subset \overline{\mathcal{G}} \ \Rightarrow \ M \in \overline{\mathcal{G}}.$$

So if $\vdash M$ then M has a normal form. Hence the system is consistent, because $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ cannot be derived.
This weak consistency result was proved by a similar method in Bunder [1983b].

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