Part III

Intersection types $\lambda\cap$ 26.2.2004:1018

Contents

In a nutshell the intersection type systems considered in this Part form a class of type assignment systems for untyped λ-calculus, extending Curry's *basic functionality* with a new type constructor, *intersection.* This simple move allows to express naturally and in a finitary way many of the most important *operational and denotational* properties of terms.

Intersection types have been originally introduced in a number of papers, Coppo and Dezani-Ciancaglini [1980], Coppo et al. [1980] and Barendregt et al. [1983], as a language for describing and capturing properties of λ-terms, which had escaped all previous typing disciplines. For instance, they were used in Pottinger [1980] in order to give the first type theoretic characterization of *strongly normalizable* terms, and later in Coppo et al. [1987] in order to capture *persistently normalizing terms* and *normalizing terms.*

Very early on it was realized, that intersection types had also a very distinctive semantical flavour. Namely, they expressed at a syntactical level the fact that a term belonged to suitable compact open sets in a Scott domain. Building on this intuition, intersection types were used in Barendregt et al. [1983] to give a proof of the completeness of the natural semantics of simple type assignment systems in applicative structures suggested in Scott [1972].

Since then, intersection types have been used as a powerful tool both for the analysis and the synthesis of λ-models, see *e.g.* Barendregt et al. [1983], Coppo et al. [1984], Coppo et al. [1987], Alessi [1991], Egidi et al. [1992], Honsell and Ronchi Della Rocca [1992], Di Gianantonio and Honsell [1993], Plotkin [1993], Honsell and Lenisa [1999], Pravato et al. [n.d.]. On the one hand, intersection type disciplines provide finitary inductive definitions of interpretation of λ-terms in models. On the other hand, they are suggestive for the shape the domain model has to have in order to exhibit certain properties, Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992].

Intersection types can be viewed also as a restriction of the domain theory in logical form, see Abramsky [1991], to the special case of modeling pure lambda calculus by means of ω-algebraic complete lattices. Many properties of these models can be proved using this paradigm, which goes back to Stone duality.

Chapter 13

An exemplary system 26.2.2004:1018

There are several variants of systems for intersection type assignment. This family wil be collectively denoted by $\lambda\cap$. In this section we consider one particular system of this family in order to outline the concepts and related properties. Definitions and the statement of theorems will be given, but no proofs. The proofs can be found in the next chapters of Part III.

13.1. The system of type assignment $\lambda\cap^{\mathbf{BCD}}$

As typical member of the family of intersection type assignment systems is $\lambda\cap^{\mathrm{BCD}}$. This system is introduced in Barendregt et al. [1983] as an extension of the initial system in Coppo and Dezani-Ciancaglini [1980].

13.1.1. Definition. (i) Let $\mathbb{C} = \mathbb{C}_\infty \cup \{\Omega\}$, where $\mathbb{C}_\infty$ is an countably infinite set of atoms and $\Omega \notin \mathbb{C}_\infty$.

(ii) The *intersection type language* over $\mathbb{C}$, denoted by $\mathbb{T} = \mathbb{T}_\cap(\mathbb{C})$ is defined by the following abstract syntax.

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T}{\to}\mathbb{T} \mid \mathbb{T} \cap \mathbb{T}$$

13.1.2. Remark. In Part III the set of syntactic types $\mathbb{T}$ will be formed as above; for many of these systems the set $\mathbb{C}$ will be finite.

13.1.3. Definition (Intersection type preorder). On $\mathbb{T}$ a binary relation $\leq$ is defined by the following axioms and rules.

(refl)	$A \leq A$
(incl_L)	$A \cap B \leq A$
(incl_R)	$A \cap B \leq B$
(glb)	$\dfrac{C \leq A \quad C \leq B}{C \leq A \cap B}$
(trans)	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$
(Ω)	$A \leq \Omega$
($\Omega\eta$)	$\Omega \leq \Omega{\to}\Omega$
($\to$-Ω)	$(A{\to}B) \cap (A{\to}C) \leq A{\to}(B \cap C)$
(η)	$\dfrac{A' \leq A \quad B \leq B'}{(A{\to}B) \leq (A'{\to}B')}$

13.1.4. Definition. The intersection type theory ∇^{BCD} is the set of all judgements $A \leq B$ derivable from the axioms and rules in definition 13.1.3.

13.1.5. Remark. All systems in Part III have the first five axioms and rules of 13.1.3. They differ in the extra rules and the set of constants.

13.1.6. Definition. We write $A \sim B$ for $A \leq B$ & $B \leq A$. In Σ we work with $\mathbb{T}$ modulo $\sim$. Notice that by rule (η) one has

$$A \sim A' \;\&\; B \sim B' \;\Rightarrow\; (A{\to}B) \sim (A' \sim B').$$

13.1.7. Definition. (i) A *basis* is a (possibly infinite) set of statements of the shape $x{:}B$, where $B{\in}\mathbb{T}$, with all variables distinct.

(ii) The type assignment system $\lambda\cap^{\text{BCD}\Omega}$ for deriving statements of the form $\Gamma \vdash M : A$ with Γ a basis, $M{\in}\Lambda$ (the set of untyped lambda terms) and $A{\in}\mathbb{T}$ is defined by the following axioms and rules.

(Ax)	$\Gamma \vdash x{:}A$	if $(x{:}A){\in}\Gamma$
($\to$I)	$\dfrac{\Gamma,\, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A{\to}B)}$	
($\to$E)	$\dfrac{\Gamma \vdash M : (A \to B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B}$	
($\cap$I)	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : (A \cap B)}$	
($\leq$)	$\dfrac{\Gamma \vdash M : A}{\Gamma \vdash M : B}$	if $A \leq B$
(Ω-universal)	$\Gamma \vdash M : \Omega$	

(iii) We say that a term M is *typable* for a given basis Γ, if there is a type $A\in\mathbb{T}$ such that the judgement $\Gamma \vdash M : A$ is derivable in $\lambda\cap^{\mathrm{BCD}\Omega}$. In this case we write $\Gamma \vdash^{\mathrm{BCD}\Omega} M : A$ or just $\Gamma \vdash M : A$ if there is little danger of confusion.

Notice that the rules $(\cap\mathrm{E})$

$$\frac{\Gamma \vdash M : (A \cap B)}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : (A \cap B)}{\Gamma \vdash M : B}$$

are derivable in $\lambda\cap^{\mathrm{BCD}\Omega}$.

13.1.8. REMARK. All systems in Part III have the first five axioms and rules of definition 13.1.7.

13.1.9. PROPOSITION. *The following rules are admissible*[1] *in the intersection type assignment system* $\lambda\cap^{BCD\Omega}$.

(*weakening*)	$\dfrac{\Gamma \vdash M : A \quad x \notin \Gamma}{\Gamma, x{:}B \vdash M : A};$
(*strengthening*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad x \notin FV(M)}{\Gamma \vdash M : A};$
(*cut*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M[x := N]) : A};$
($\leq$-*L*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad C \leq B}{\Gamma, x{:}C \vdash M : A};$
($\to$L)	$\dfrac{\Gamma, y{:}B \vdash M : A \quad \Gamma \vdash N : C \quad x \notin \Gamma}{\Gamma, x{:}(C\to B) \vdash (M[y := xN]) : A};$
($\cap$L)	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma, x{:}(A \cap C) \vdash M : B}.$

13.1.10. THEOREM. (i) *Let* $A \not\sim \Omega$. *Then*

$$\Gamma \vdash x : A \;\Leftrightarrow\; \exists B\in\mathbb{T}.[(x{:}B\in\Gamma \;\&\; B \leq A].$$

(ii) $\Gamma \vdash (MN) : A \;\Leftrightarrow\; \exists B\in\mathbb{T}.[\Gamma \vdash M : (B\to A) \;\&\; \Gamma \vdash N : B]$.

(iii) $\Gamma \vdash \lambda x.M : A \;\Leftrightarrow\; \exists B_i, C_i\in\mathbb{T}.[\Gamma, x{:}B_i \vdash M : C_i \;\&\; \bigcap_i(B_i\to C_i) \leq A]$

(iv) $\Gamma \vdash \lambda x.M : B\to C \;\Leftrightarrow\; \Gamma, x{:}B \vdash M : C$.

13.1.11. DEFINITION. Let R be a notion of reduction. We introduce the following rules

(R-red)	$\dfrac{\Gamma \vdash M : A \quad M \to_R N}{\Gamma \vdash N : A}$
(R-exp)	$\dfrac{\Gamma \vdash M : A \quad M \leftarrow_R N}{\Gamma \vdash N : A}$

[1] This means that e.g. $\Gamma \vdash M : A \quad x \notin \Gamma \;\Rightarrow\; \Gamma, x{:}B \vdash M : A$ is valid.

13.1.12. PROPOSITION. *The rules (β-red), (β-exp) and (η-red) are admissible in* $\lambda\cap^{BCD\Omega}$. *The rule (η-exp) is not.*

The following result characterizes notions related to normalization in terms of type assignment in system $\lambda\cap^{\mathrm{BCD}\Omega}$. The notation $\Omega \notin A$ means that Ω is not a subtype of A. The system $\lambda\cap^{\mathrm{BCD}}$ is the system $\lambda\cap^{\mathrm{BCD}\Omega}$ in which the axiom (Ω-universal) is left out. As the system with Ω is the default, we write $\vdash$ for type assignment in $\lambda\cap^{\mathrm{BCD}\Omega}$ and $\vdash^{-\Omega}$ for type assignment in $\lambda\cap^{\mathrm{BCD}}$.

13.1.13. THEOREM. *Let* $M{\in}\Lambda^{\emptyset}$.

(i) *M has a head normal form* $\Leftrightarrow$ $\exists A{\in}\mathbb{T}.[A \not\sim \Omega \;\&\; \vdash M : A]$.

(ii) *M has a normal form* $\Leftrightarrow$ $\exists A{\in}\mathbb{T}.[\Omega \notin A \;\&\; \vdash M : A]$.

(iii) *M is strongly normalizing* $\Leftrightarrow$ $\exists A{\in}\mathbb{T}. \vdash^{-\Omega} M : A$.

The set of types of a term coincides with the set of types of its approximants: this will give an Approximation Theorem for the filter model of next section.

13.1.14. THEOREM. $\Gamma \vdash M : A$ *iff there is an approximant* P *of* M *such that* $\Gamma \vdash P : A$.

Finally the question whether an intersection type is inhabited is undecidable.

13.1.15. THEOREM. *The set* $\{A{\in}\mathbb{T} \mid \exists M{\in}\Lambda^{\emptyset} \vdash M : A\}$ *is undecidable.*

13.2. The filter model

13.2.1. DEFINITION. (i) A *complete lattice* $(\mathcal{D}, \sqsubseteq)$ is a partial order which has arbitrary least upper bounds (sup's) (and hence has arbitrary inf's).

(ii) A subset $Z \subseteq \mathcal{D}$ is *directed* if $Z \neq \emptyset$ and

$$\forall x, y{\in}Z \exists z{\in}Z. x, y \sqsubseteq z.$$

(iii) An element $c{\in}\mathcal{D}$ is *compact* (also called *finite*) if for each directed $Z \subseteq \mathcal{D}$ one has

$$c \sqsubseteq \bigsqcup Z \;\Rightarrow\; \exists z{\in}Z. c \sqsubseteq z.$$

Let $\mathcal{K}(\mathcal{D})$ denote the set of finite elements of $\mathcal{D}$.

(iv) A complete lattice is ω-*algebraic* if $\mathcal{K}(\mathcal{D})$ is countable, and for each $d{\in}D$, the set $\mathcal{K}(d) = \{c{\in}\mathcal{K}(\mathcal{D}) \mid c \sqsubseteq d\}$ is directed and $d = \bigsqcup \mathcal{K}(d)$.

(v) Let $(\mathcal{D}, \sqsubseteq)$ be an ω-*algebraic* complete lattice. The Scott topology on $\mathcal{D}$ contains as open sets the $U \subseteq \mathcal{D}$ such that

(1) $d{\in}U \;\&\; d \sqsubseteq e \;\Rightarrow\; e{\in}U$;

(2) if $Z \subseteq \mathcal{D}$ is directed then $\bigsqcup Z{\in}U \;\Rightarrow\; \exists z{\in}Z. z{\in}U$.

(vi) If $\mathcal{D}, \mathcal{E}$ are ω-algebraic complete lattices, then $[\mathcal{D}{\to}\mathcal{E}]$ denotes the set of continuous maps from $\mathcal{D}$ to $\mathcal{E}$. This set can be ordered pointwise

$$f \sqsubseteq g \;\Leftrightarrow\; \forall d{\in}\mathcal{D}. f(d) \sqsubseteq g(d)$$

and $\langle \mathcal{D}, \sqsubseteq \rangle$ is again an ω-algebraic lattice. (If $d{\in}\mathcal{K}(\mathcal{D}), e{\in}\mathcal{K}(\mathcal{E})$, then the step function $f_{d,e}$ defined by $f_{d,e}(x) = $ if $d \sqsubseteq x$ then e else $\perp$ is in $\mathcal{K}([\mathcal{D}{\to}\mathcal{E}])$.)

(vii) The category ω-**ALG** is the category whose objects are the ω-algebraic complete lattices and whose morphisms are the (Scott) continuous functions.

13.2.2. DEFINITION. (i) A *filter* over $\mathbb{T}$ is a set $X \subseteq \mathbb{T}$ such that

(1) $\Omega \in X$;

(2) $A \in X \ \& \ A \leq B \ \Rightarrow \ B \in X$;

(3) $A, B \in X \ \Rightarrow \ (A \cap B) \in X$.

(ii) $\mathcal{F}$ denotes the set of filters over $\mathbb{T}$.

(iii) If $X \subseteq \mathbb{T}$, then $\uparrow X$ is

$$\{A \mid \exists B_1 \ldots B_n \in X . B_1 \cap \ldots \cap B_n \leq A\}$$

is the filter *generated by* X.

(iv) A *principal* filter is of the form $\uparrow\{A\}$ for some $A \in \mathbb{T}$. We shall denote this simply by $\uparrow A$. Note that $\uparrow A = \{B \mid A \leq B\}$.

13.2.3. PROPOSITION. (i) *$\mathcal{F} = \langle \mathcal{F}, \subseteq \rangle$ is an ω-algebraic complete lattice.*

(ii) *$\mathcal{F}$ has as bottom element $\uparrow\Omega$ and as top element $\mathbb{T}$.*

(iii) *The finite elements of $\mathcal{F}$ are exactly the principal filters.*

13.2.4. DEFINITION. (i) A *λ-structure* is a triple $\langle \mathcal{D}, F, G \rangle$, where $\mathcal{D}$ is an ω-algebraic lattice and $F : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$, $G : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}$ are Scott continuous.

(ii) If moreover $F \circ G = \mathtt{id}_{[\mathcal{D} \to \mathcal{D}]}$, then $\langle \mathcal{D}, F, G \rangle$ is called a *retract*.

13.2.5. PROPOSITION. *Define maps $F : \mathcal{F} \to (\mathcal{F} \to \mathcal{F})$ and $G : [\mathcal{F} \to \mathcal{F}] \to \mathcal{F}$ by*

$$\begin{array}{rcl} F(X)(Y) & = & \uparrow\{B \mid \exists A \in Y . (A \to B) \in X\} \\ G(f) & = & \uparrow\{A \to B \mid B \in f(\uparrow A)\}. \end{array}$$

Then $\langle \mathcal{D}, F, G \rangle$ is a retract.

Being $\langle \mathcal{D}, F, G \rangle$ a retract, it is also a λ-model in which the term interpretation is naturally defined as follows (see Barendregt [1984], Section 5.4).

13.2.6. DEFINITION (Interpretation of terms). (i) A *term environment* in $\mathcal{F}$ is a map $\rho : \mathtt{Var} \to \mathcal{F}$.

(ii) If ρ is a term environment and $d \in \mathcal{F}$, then $\rho(x := d)$ is the term environment ρ' defined by

$$\begin{array}{rcll} \rho'(y) & = & \rho(y) & \text{if } y \not\equiv x; \\ \rho'(x) & = & d. & \end{array}$$

(iii) Given a term environment ρ, the interpretation $[\![\ ]\!]_\rho : \Lambda \to \mathcal{F}$ is defined as follows.

$$\begin{array}{rcl} [\![x]\!]_\rho & = & \rho(x); \\ [\![MN]\!]_\rho & = & F[\![M]\!]_\rho [\![N]\!]_\rho; \\ [\![\lambda x.M]\!]_\rho & = & G(\boldsymbol{\lambda} d \in \mathcal{F} . [\![M]\!]_{\rho(x:=d)}). \end{array}$$

An important property of the λ-model $\langle \mathcal{D}, F, G\rangle$ is that the meaning of a term is the set of types which are deducible for it.

13.2.7. THEOREM.

$$[\![M]\!]_\rho \quad = \quad \{A \mid \exists\Gamma \models \rho.\Gamma \vdash M : A\},$$

where $\Gamma \models \rho$ *iff* $(x{:}B)\in\Gamma \;\Rightarrow\; B\in\rho(x)$.

Lastly we notice that all continous functions are representable.

13.2.8. THEOREM.

$$[\mathcal{F}\to\mathcal{F}] = \{f : \mathcal{F}\to\mathcal{F} \mid f \textit{ is representable}\},$$

where $f\in\mathcal{F}\to\mathcal{F}$ *is called* representable *iff for some* $X\in\mathcal{F}$ *one has* $f(Y) = F(X)(Y)$ *for all* $Y\in\mathcal{F}$.

13.3. Completeness of type assignment

13.3.1. DEFINITION (Interpretation of types). Let $\mathcal{M} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{M}}\rangle$ be a λ-model.

(i) A *type environment* in $\mathcal{D}$ is a map $\mathcal{V} : \mathbb{C}_\infty \to \mathcal{P}(\mathcal{D})$.

(ii) For $X, Y\in\mathcal{P}(\mathcal{D})$ define

$$X\to Y = \{d\in\mathcal{D} \mid d.X \subseteq Y\} = \{d\in\mathcal{D} \mid \forall x\in X.d.x\in Y\}.$$

(iii) Given a type environment $\mathcal{V}$, the interpretation $[\![\ ]\!]_\mathcal{V} : \mathbb{T}\to\mathcal{P}(\mathcal{D})$ is defined as follows.

$$\begin{array}{rcl}
[\![\Omega]\!]_\mathcal{V} & = & \mathcal{D}; \\
[\![\varphi]\!]_\mathcal{V} & = & \mathcal{V}(\varphi); \\
[\![A\to B]\!]_\mathcal{V} & = & [\![A]\!]_\mathcal{V}\to[\![B]\!]_\mathcal{V}; \\
[\![A\cap B]\!]_\mathcal{V} & = & [\![A]\!]_\mathcal{V}\cap[\![B]\!]_\mathcal{V}.
\end{array}$$

13.3.2. DEFINITION (Satisfaction). Given a term environment ρ and a type environment $\mathcal{V}$ one defines the following.

(i) $\rho, \mathcal{V} \models M : A \;\Leftrightarrow\; [\![M]\!]^{\mathcal{M}}_\rho\in[\![A]\!]_\mathcal{V}$ for all $\mathcal{M}$.

(ii) $\rho, \mathcal{V} \models \Gamma \;\Leftrightarrow\; \rho, \mathcal{V} \models x : B$ for all $(x{:}B)\in\Gamma$.

(iii) $\Gamma \models M : A \;\Leftrightarrow\; \forall\rho, \mathcal{V}.[\rho, \mathcal{V} \models \Gamma \;\Rightarrow\; \rho, \mathcal{V} \models M : A]$.

13.3.3. THEOREM (Soundness).

$$\Gamma \vdash M : A \;\Rightarrow\; \Gamma \models M : A.$$

13.3.4. THEOREM (Completeness).

$$\Gamma \models M : A \;\Rightarrow\; \Gamma \vdash M : A.$$

Chapter 14

The systems $\lambda\cap$ 26.2.2004:1018

There are various alternative, but ultimately equivalent, presentations of intersection type assignment systems. We shall mainly focus on a natural deduction style version, because this is most commonly used. Occasionaly we shall discuss alternatives.

Intersection types are syntactic objects forming an algebra $\mathbb{T}$ freely generated from a set of atoms $\mathbb{C}$ by means of the operators $\rightarrow$ and $\cap$. Using some set $\triangledown$ of axioms and rules a preorder $\leq_{\triangledown}$ is defined on $\mathbb{T}$. Working modulo the corresponding equivalence relation $\sim_{\triangledown}$ one obtains a meet semi-lattice (poset with meet operator) $\langle\mathbb{T}/\sim_{\triangledown},\leq,\cap\rangle$. For almost all $\triangledown$ the operator $\rightarrow$ preserves $\sim_{\triangledown}$ and one obtains a type structure $\langle\mathbb{T}/\sim_{\triangledown},\rightarrow,\leq,\cap\rangle$.

Given $\triangledown$ (or just the type structure) one defines a type assignment system $\lambda\cap^{\triangledown}$. For these type assignment systems the type reconstruction problem is in general undecidable. Using $\triangledown$ one can semantically characterize several sets of lambda terms, e.g. the (strongly) normalizing ones. Using $\triangledown$ one also defines corresponding *filter λ-models* $\mathcal{F}^{\triangledown}$. It turns out that several classical and new models for untyped λ-calculus can be obtained in this way. Using approximation theorems for the type assignment systems the local structure of the models will be studied.

We introduce the systems and its basic properties in this Chapter. More precisely, intersection type languages and intersection type theories are introduced in Section 14.1. In Section 14.2 the corresponding type assignment systems are defined.

14.1. Intersection type structures and theories 26.2.2004:1018

14.1.1. DEFINITION. (i) A *type structure* (TS) is a type algebra $\mathcal{T}$, see definition **??**, with a partial order on it

$$\mathcal{T}=\langle|\mathcal{T}|,\rightsquigarrow,\leq\rangle.$$

As in Part II we write $A{\in}\mathcal{T}$ for $A{\in}|\mathcal{T}|$.

(ii) An *intersection type structure* (ITS) is a type structure that is also a meet semi-lattice: every $A,B{\in}\mathcal{T}$ have a greatest lower bound (glb) $A\cap B{\in}\mathcal{T}$.

Intersection Type theories

Often type structures are *syntactic ones*, i.e. consisting of syntactic types modulo some theory.

14.1.2. DEFINITION. The set of *intersection types* over $\mathbb{C}$, notation $\mathbb{T} = \mathbb{T}_\cap(\mathbb{C})$, is defined by the following abstract syntax

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T}{\to}\mathbb{T} \mid \mathbb{T} \cap \mathbb{T}.$$

14.1.3. DEFINITION. (i) A *type theory* (TT) is a set of statements $\triangledown$ of the form $A \leq B$ with $A, B{\in}\mathbb{T}_\cap(\mathbb{C})$ satisfying the following axiom and rule

$$\boxed{\begin{array}{ll} (\text{refl}) & A \leq A \\[1ex] (\text{trans}) & \dfrac{A \leq B \quad B \leq C}{A \leq C} \end{array}},$$

i.e. $(A \leq A){\in}\nabla$ and $(A \leq B), (B \leq C){\in}\nabla \;\Rightarrow\; (A \leq C){\in}\nabla$, for all A, B, C.

(ii) An *intersection type theory* (ITT) is a type theory $\triangledown$ satisfying also

$$\boxed{\begin{array}{ll} (\text{incl}_L) & A \cap B \leq A \\[1ex] (\text{incl}_R) & A \cap B \leq B \\[1ex] (\text{glb}) & \dfrac{C \leq A \quad C \leq B}{C \leq A \cap B} \end{array}}.$$

(iii) Since several (intersection) type theories will be considered we use the following notation. For an intersection type theory $\triangledown$ write

$$A \leq_\triangledown B \;\Leftrightarrow\; (A \leq B){\in}\triangledown$$

and

$$A{\sim_\triangledown}B \;\Leftrightarrow\; A \leq_\triangledown B \leq_\triangledown A.$$

If there is little danger of confusion and ∇ is clear from the context, then we will write $\leq, \sim$ for $\leq_\triangledown, \sim_\triangledown$.

(iv) An (intersection) type theory $\triangledown$ is called *extensional* if it satisfies the following rule

$$\boxed{\begin{array}{ll} (\eta^\sim) & \dfrac{A \sim A' \quad B \sim B'}{(A{\to}B) \sim (A'{\to}B')} \end{array}}.$$

14.1.4. REMARK. The rule (glb) implies

$$\boxed{\begin{array}{ll} (\text{mon}) & \dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'} \end{array}}.$$

NOTATION. Upper case Roman letters i.e. $A, B, \ldots$, will denote arbitrary types (either syntactic or elements of a type structure). When writing intersection types we shall use the following convention: the constructor $\cap$ takes precedence over the constructor $\to$ and it associates to the right. For example

$$(A{\to}B{\to}C) \cap A{\to}B{\to}C \equiv ((A{\to}(B{\to}C)) \cap A){\to}(B{\to}C).$$

14.1.5. PROPOSITION. *Let $\triangledown$ be an extensional type theory.*

(i) *Then $\triangledown$ induces a type structure*

$$\mathcal{T}^{\triangledown} = \langle \mathbb{T}_{\cap}(\mathbb{C})/{\sim_{\triangledown}}, \leadsto, \leq \rangle,$$

by defining on the $\sim_{\triangledown}$-equivalence classes

$$\begin{array}{rcl} [A] \leadsto [B] & = & [A{\to}B]; \\ [A] \leq [B] & \Leftrightarrow & A \leq_{\triangledown} B. \end{array}$$

(ii) *If $\triangledown$ is moreover an intersection type theory, then $\mathcal{T}^{\triangledown}$ is an ITS.*

PROOF. Easy. Extensionality is needed to ensure that $\leadsto$ is well-defined. Note that in case ∇ is an ITT, then $[A \cap B]$ is the glb of $[A], [B]$. ∎

The (I)TS $\mathcal{T}^{\nabla}$, with ∇ an extensional (I)TT, is called a *syntactical* (I)TS.

14.1.6. PROPOSITION. *Every (I)TS $\mathcal{T}$ can be considered as a syntactical one.*

PROOF. For a type structure $\mathcal{T}$, define ∇ by taking $\mathbb{C} = \{\psi_c \mid c{\in}\mathcal{T}\}$ and the axiomatization (refl), (trans), $(\eta^{\sim})$, and

$$\frac{a \leq_{\mathcal{T}} b}{\psi_a \leq_{\triangledown} \psi_b};$$
$$(\psi_a{\to}\psi_b){\sim_{\triangledown}}\psi_{a\to b}.$$

For an intersection type structure, also add to the axiomatization of ∇ the rules (incl_L), (incl_R) and (glb) and $(\psi_a \cap \psi_b){\sim_{\triangledown}}\psi_{a\cap b}$. In both cases $\mathcal{T} \cong \mathcal{T}^{\nabla}$, as can be seen as follows. Define $f : \mathcal{T}{\to}\mathcal{T}^{\nabla}$ by $f(a) = [\psi_a]$. For the inverse, first define $g : \mathbb{T}_{\cap}(\mathbb{C}){\to}\mathcal{T}$ by

$$\begin{array}{rcl} g(\psi_c) & = & c; \\ g(A{\to}B) & = & g(A){\to}g(B); \\ g(A \cap B) & = & g(A) \cap g(B). \end{array}$$

Then show $A \leq_{\triangledown} B \Rightarrow g(A) \leq g(B)$. Finally set $f^{-1}([A]) = g(A)$, which is well defined. It is easy to show that f, f^{-1} constitute an isomorphism. ∎

Although essentially equivalent, type structures and extensional type theories differ in the following. In the theories the types are freely generated from a fixed set of constants and the inequality can be controlled somewhat by choosing the right axioms and rules (this will be exploited in definition 17.3.9). In type structures one has the antisymmetric law $A \leq B \leq A \Rightarrow A = B$, which is in line with the common theory of partial orders (this will be exploited in Proposition 16.2.21).

Specific Intersection Type theories

Now we will construct several syntactic intersection type structures, that will play an important role in later chapters, by introducing the following axiom schemes, rule schemes and axioms. The constants that will be used in these belong to the set $\mathbb{C} = \mathbb{C}_\infty \cup \{\Omega, \nu, \varphi, \omega\}$, where $\mathbb{C}_\infty = \{\varphi_0, \varphi_1, \ldots\}$ is an infinite set of fresh (i.e. different from $\Omega, \nu, \varphi, \omega$) constants. The $A, B, C, \ldots$ range over $\mathbb{T}_\cap(\mathbb{C})$.

14.1.7. NOTATION. We will consider several type structures and theories with top element. If in a type theory we use Ω, then this constant always denotes the top, i.e. satisfies for all A

$$(\Omega) \quad A \leq \Omega.$$

In most type theories this is how the top will be denoted. In some other cases of ITTs there is a type that is the top, but this needs to be proved[1].

14.1.8. NOTATION. We introduce names for axiom(scheme)s and rule(scheme)s in the following table.

(Ω)	$A \leq \Omega$
(ν)	$(A{\to}B) \leq \nu$
(Ω_{lazy})	$(A{\to}B) \leq (\Omega{\to}\Omega)$
$({\to}\cap)$	$(A{\to}B) \cap (A{\to}C) \leq A{\to}B \cap C$
$({\to}\cap^{\sim})$	$(A{\to}B) \cap (A{\to}C) \sim A{\to}B \cap C$
$(\Omega\eta)$	$\Omega \leq (\Omega{\to}\Omega)$
(ω_{Scott})	$(\Omega{\to}\omega) \sim \omega$
(ω_{Park})	$(\omega{\to}\omega) \sim \omega$
$(\omega\varphi)$	$\omega \leq \varphi$
$(\varphi{\to}\omega)$	$(\varphi{\to}\omega) \sim \omega$
$(\omega{\to}\varphi)$	$(\omega{\to}\varphi) \sim \varphi$
(I)	$(\varphi{\to}\varphi) \cap (\omega{\to}\omega) \sim \varphi$
(η)	$\dfrac{A' \leq A \quad B \leq B'}{(A{\to}B) \leq (A'{\to}B')}$
$(\eta^{\sim})$	$\dfrac{A' \sim A \quad B \sim B'}{(A{\to}B) \sim (A'{\to}B')}$

Figure 14.1: Possible Axioms and Rules concerning $\leq$.

[1] This happens e.g. in the theory ∇ = HL, where there is an element φ which is an *a posteriori* top: there is no axiom $A \leq \varphi$, but one can derive that φ is the top type of $\mathcal{T}^{\text{HL}}$, see Exercise 14.3.1(i). Also in ∇ = EHR one has the type ν as an *a posteriori* top, see Exercise 14.3.24(i).

We now introduce a list of specific ITTs which have been extensively considered in the literature. The order is logical, rather than historical, and some of the references define the models directly, others deal with the corresponding filter models (see Sections 16 and 17.3): Coppo and Dezani-Ciancaglini [1980], Coppo et al. [1981], Honsell and Lenisa [1999], Egidi et al. [1992], Abramsky and Ong [1993], Barendregt et al. [1983], Scott [1972], Park [1976], Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992], **?**, Plotkin [1993], Engeler [1981].

We shall denote the theories $\triangledown$ by names (respectively acronymes) of the author(s) who have first considered the λ-model induced by such a theory. In Figure 14.2 we specify each theory by giving a set of constants $\mathbb{C}^{\triangledown}$ and axioms and rules, to be found in Figure 14.1.8.

14.1.9. Definition. We define the following intersection type theories $\triangledown$ by specifying the set $\mathbb{C}^{\triangledown}$, the rule- and axiom schemes and the special axioms.

$\triangledown$	$\mathbb{C}^{\triangledown}$	Rules	Axioms	Special axioms
Scott	$\{\Omega,\omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta),(\omega_{\text{Scott}})$
Park	$\{\Omega,\omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta),(\omega_{\text{Park}})$
CDZ	$\{\Omega,\varphi,\omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta),(\omega\varphi),(\varphi{\to}\omega),(\omega{\to}\varphi)$
HR	$\{\Omega,\varphi,\omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta),(\omega\varphi),(\varphi{\to}\omega),(I)$
DHM	$\{\Omega,\varphi,\omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta),(\omega\varphi),(\omega{\to}\varphi),(\omega_{\text{Scott}})$
BCD	$\{\Omega\}\cup\mathbb{C}_\infty$	(η)	$(\to\cap)$	$(\Omega),(\Omega\eta)$
AO	$\{\Omega\}$	(η)	$(\to\cap)$	$(\Omega),(\Omega_{\text{lazy}})$
HL	$\{\varphi,\omega\}$	(η)	$(\to\cap)$	$(\omega\varphi),(\varphi{\to}\omega),(\omega{\to}\varphi)$
CDV	$\mathbb{C}_\infty$	(η)	$(\to\cap)$	–
Engeler	$\{\Omega\}\cup\mathbb{C}_\infty$	$(\eta^\sim)$	$(\to\cap)$	$(\Omega),(\Omega\eta)$
Plotkin	$\{\Omega,\varphi\}$	$(\eta^\sim)$	–	(Ω)
EHR	$\{\nu\}$	(η)	–	(ν)
CD	$\mathbb{C}_\infty$	–	–	–

Figure 14.2: Various type theories

Historically the systems CD, CDV and BCD were among the early ones.

The expressive power of intersection types is remarkable. This will become apparent when we will use them as a tool for characterizing properties of λ-terms (see Sections 17.1 and 16.5), and for describing different λ-models (see Section 17.3). Much of this expressive power comes from the fact that they are endowed with a *preorder relation*, $\leq$, which induces, on the set of types, the structure of a meet semi-lattice with respect to $\cap$. This appears natural when we think of types as subsets of a domain of discourse D, which is endowed with a (partial) application $\cdot : D\times D{\to}D$, and interpret $\cap$ as set-theoretic intersection, $\leq$ as set inclusion, and give $\to$ the *realizability interpretation*.

$$\begin{array}{rcl} [\![A]\!] & \subseteq & D \\ A\leq B & \Leftrightarrow & [\![A]\!]\subseteq[\![B]\!] \\ [\![A\cap B]\!] & = & [\![A]\!]\cap[\![B]\!] \\ [\![A{\to}B]\!] & = & [\![A]\!]{\to}[\![B]\!] = \{d{\in}D \mid d\cdot[\![A]\!]\subseteq[\![B]\!]\}. \end{array}$$

This semantics, due to Scott, will be studied in chapter 16.

In Fig. 14.2 the axiom (Ω) states that the type structure $\mathcal{T}$ has a top element. This is particularly meaningful when used in combination with the $\lambda\cap^{\mathcal{T}\Omega}$-type assignment system, which essentially treats Ω as the universal type of all λ-terms (see Definition 14.2.2(ii)).

If Ω is the top element, then $\Omega\to\Omega$ is the set of functions which applied to an arbitrary element return again an arbitrary element. In that case axiom ($\Omega\eta$) expresses the fact that all the objects in our domain of discourse are total functions, i.e. that Ω is equal to $\Omega\to\Omega$ (Barendregt et al. [1983]). If now we want to capture only those terms which truly represent functions, as we do for example in the lazy λ-calculus, we cannot assume axiom ($\Omega\eta$). One still may postulate the weaker property (Ω_{lazy}) to make all functions total (Abramsky and Ong [1993]). It simply says that an element which is a function, because it maps A into B, maps also the whole universe into itself.

The intended interpretation of arrow types also motivates axiom ($\to\cap$), which implies that if a function maps A into B, and the same function maps also A into C, then, actually, it maps the whole A into the intersection between B and C (i.e. into $B\cap C$), see Barendregt et al. [1983].

Rule (η) is again very natural in view of the set-theoretic interpretation. It implies that the arrow constructor is contravariant in the first argument and covariant in the second one. It is clear that if a function maps A into B, and we take a subset A' of A and a superset B' of B, then this function will map also A' into B', see Barendregt et al. [1983].

The rules ($\to$-$\cap^\sim$) and ($\eta^\sim$) are similar to the rules ($\to$-$\cap$) and (η). They capture properties of the graph models for the untyped lambda calculus, see Plotkin [1975] and Engeler [1981], as we shall discuss in Section 17.3.

The remaining axioms express peculiar properties of D_∞-like inverse limit models, see Barendregt et al. [1983], Coppo et al. [1984], Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992], Honsell and Lenisa [1993]. We shall discuss them more in detail in Section 17.3.

Properties of intersection type structures

The first properties are concerned with concrete ITTs.

14.1.10. LEMMA. (i) *Let*

$$\nabla\in\{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{BCD}, \mathit{Engeler}, \mathit{Plotkin}\}$$

be one of the ITTs defined in 14.2. Define inductively the following collection of types

$$\boxed{\mathcal{E} \;:=\; \Omega \mid \mathbb{T}^\nabla\to\mathcal{E} \mid \mathcal{E}\cap\mathcal{E}}$$

Then $A\leq B, A\in\mathcal{E} \;\Rightarrow\; B\in\mathcal{E}$.

(ii) *For* $\nabla\in\{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}\}$ *one has* $\omega\not\leq_\nabla\Omega$.

(iii) *For* $\nabla\in\{\mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{Plotkin}\}$ *one has* $\varphi\not\leq_\nabla\Omega$.

(iv) *For* $\nabla\in\{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{BCD}, \mathit{Engeler}, \mathit{Plotkin}\}$ *one has*

$$A\not\leq\Omega \;\Rightarrow\; \vec{B}\to A\not\leq_\nabla\Omega.$$

PROOF. (i) By induction on the generation of $A \leq B$.
(ii) (iii) and (iv) Immediately by (i). ■

14.1.11. LEMMA. *Define inductively*

$$\boxed{\mathcal{E} \ := \ \Omega \mid \mathcal{E} \cap \mathcal{E}}$$

Then
(i) $A \sim \Omega \ \Leftrightarrow \ A{\in}\mathcal{E}$.
(ii) $(A{\to}B) \not\sim \Omega$.

PROOF. Easy. ■

14.1.12. LEMMA. *For* $\nabla{\in}\{CDZ, HR, DHM, HL\}$ *one has* $\varphi \not\leq_\nabla \omega$.

PROOF. Case $\nabla{\in}\{\mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}\}$. Define by mutual induction

$$\boxed{\begin{array}{rcl} L_\omega & := & \omega \mid \Gamma_\varphi{\to}L_\omega \mid L_\omega \cap \mathbb{T}^\nabla \mid \mathbb{T}^\nabla \cap L_\omega \\ G_\varphi & := & \varphi \mid \Omega \mid \mathbb{T}^\nabla{\to}G_\varphi \mid L_\omega{\to}\mathbb{T}^\nabla \mid G_\varphi \cap G_\varphi \end{array}}$$

One easily shows by mutual induction

$$\begin{array}{rcl} A \leq B, B{\in}L_\omega & \Rightarrow & A{\in}L_\omega \\ A \leq B, A{\in}G_\varphi & \Rightarrow & B{\in}G_\varphi \end{array}$$

Then $\varphi \not\leq \omega$ follows immediately.
Case $\nabla = \mathrm{HL}$. Similarly, by leaving out Ω in the definition of G_φ. ■

Now we will consider some classes of ITSs.

14.1.13. DEFINITION. Let $\mathcal{T}$ be an ITS.
(i) $\mathcal{T}$ is *natural* if (Ω), $(\Omega\eta)$, $(\to\text{-}\cap)$ and (η) hold in $\mathcal{T}$ for some $\Omega{\in}\mathcal{T}$, i.e.

$$\begin{array}{lrl} (\Omega) & \forall A. & A \leq \Omega \\ (\Omega\eta) & & \Omega \leq (\Omega{\to}\Omega) \\ (\to\cap) & \forall A, B, C. & (A{\to}B) \cap (A{\to}C) \leq A{\to}B \cap C, \\ (\eta) & \forall A, A', B, B'. & A' \leq A \ \&\ B \leq B' \ \Rightarrow \ (A{\to}B) \leq (A'{\to}B') \end{array}$$

We have used the same notation for the element $\Omega{\in}\mathcal{T}$ and its name in the language of type theory.
(ii) $\mathcal{T}$ is called *restricted* natural if it satisfies $(\to\text{-}\cap)$ and (η).
(iii) $\mathcal{T}$ is called *weakly* natural if it satisfies $(\Omega), (\to\cap), (\Omega\text{-lazy})$ and (η).

For the first 7 of the 13 intersection type theories $\triangledown$ in Fig. 14.2 the structure $\mathcal{T}^\triangledown$ is natural. Moreover, $\mathcal{T}^\triangledown$ is restricted natural for the first 9 theories $\triangledown$. The type structure $\mathcal{T}^{\mathrm{HL}}$ is restricted natural, but not natural: φ is the top, but $\varphi \not\leq (\varphi{\to}\varphi)$, see exercise 14.3.1. Finally $\mathcal{T}^{\mathrm{AO}}$ is also weakly natural, but not natural, see Exercise 15.4.2(ii).

14.1.14. PROPOSITION. *Let $\mathcal{T}$ be a restricted natural ITS. Then $\mathcal{T}$ satisfies*

(i) $(A \to B) \cap (A' \to B') \leq (A \cap A') \to (B \cap B')$;

(ii) $(A_1 \to B_1) \cap \ldots \cap (A_n \to B_n) \leq (A_1 \cap \ldots \cap A_n) \to (B_1 \cap \ldots \cap B_n)$.

PROOF. (i)
$$\begin{array}{rcl}(A \to B) \cap (A' \to B') & \leq & ((A \cap A') \to B) \cap ((A \cap A') \to B') \\ & \leq & (A \cap A') \to (B \cap B'),\end{array}$$
by respectively (η) and $(\to\cap)$.

(ii) Similarly (i.e. by induction on $n>1$, using (i) for the induction step). ■

14.2. Intersection type assignment

Let $\mathcal{T}$ be an ITS. We define a type assignment system $\lambda\cap^{\mathcal{T}}$ that assigns to untyped lambda terms elements of $\mathcal{T}$ as follows.

14.2.1. DEFINITION. (i) A $\mathcal{T}$-*statement* is of the form $M : A$ with the *subject* $M \in \Lambda$ an untyped lambda term and the *predicate* $A \in \mathcal{T}$.

(ii) A *declaration* is a $\mathcal{T}$-statement of the form $x : A$.

(iii) A $\mathcal{T}$-*basis* Γ is is a set of $\mathcal{T}$-declarations, with all variables distinct.

(iv) A $\mathcal{T}$-*assertion* is of the form $\Gamma \vdash^{\nabla} M : A$, where $M : A$ is a $\mathcal{T}$-statement and Γ is a $\mathcal{T}$-basis.

14.2.2. DEFINITION. (i) Let $\mathcal{T}$ be an ITS. The *(basic) type assignment system* $\lambda\cap^{\mathcal{T}}$ derives $\mathcal{T}$-assertions $\Gamma \vdash M : A$, with $A \in \mathcal{T}$, by the following axioms and rules.

(Ax)	$\Gamma \vdash x{:}A$	if $(x{:}A \in \Gamma)$
$(\to\mathrm{I})$	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$	
$(\to\mathrm{E})$	$\dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$	
$(\leq)$	$\dfrac{\Gamma \vdash M : A \quad A \leq_{\mathcal{T}} B}{\Gamma \vdash M : B}$	
$(\cap\mathrm{I})$	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$	

Figure 14.3: Basic type assignment $\lambda\cap^{\top}$.

(ii) If $\mathcal{T}$ has a top element Ω (i.e. $A \leq \Omega$ for all A), then the Ω type assignment system $\lambda\cap^{\mathcal{T}\Omega}$ is defined by adding the extra axiom

(Ω universal)	$\Gamma \vdash M : \Omega$

Figure 14.4: Omega type assignment $\lambda\cap^{\mathcal{T}\Omega}$.

14.2.3. NOTATION. We write $\Gamma \vdash^{\mathcal{T}(\Omega)} M : A$ if $\Gamma \vdash M : A$ is derivable in $\lambda\cap^{\mathcal{T}(\Omega)}$. In case $\mathcal{T} = \mathcal{T}^{\nabla}$ we write for $x_1{:}[A_1].\ldots, x_n{:}[A_n] \vdash^{\mathcal{T}^{\nabla}(\Omega)} M : [A]$

simply $x_1{:}A_1,\ldots,x_n{:}A_n \vdash^{\nabla(\Omega)} M : A$; this is well-defined. Both $\vdash^{\mathcal{T}(\Omega)}$ and $\vdash^{\nabla(\Omega)}$ may also be written as $\vdash^{(\Omega)}$, if there is little danger of confusion. Similarly, $\lambda\cap^{\nabla(\Omega)}$ is $\lambda\cap^{\mathcal{T}_\nabla(\Omega)}$, and $\lambda\cap^{\mathcal{T}(\Omega)}$ or $\lambda\cap^{\nabla(\Omega)}$ may both be denoted by $\lambda\cap^{(\Omega)}$.

14.2.4. Example. Let $\mathcal{T}$ be an intersection type structure with top Ω and elements A, B.

(i) $\vdash^{\mathcal{T}} (\lambda x.xx) : A \cap (A{\to}B){\to}B$.
$\vdash^{\mathcal{T}\Omega} (\lambda x.xx)(\lambda x.xx) : \Omega$.

(ii) Let $M \equiv \mathsf{KI}((\lambda x.xx)(\lambda x.xx))$. Then $\vdash^{\mathcal{T}\Omega} M : (A{\to}A)$, but in general $\not\vdash^{\mathcal{T}} M : (A{\to}A)$.

(iii) (van Bakel) Let $M \equiv \lambda yz.\mathsf{K}z(yz)$ and $N \equiv \lambda yz.z$. Then $M \twoheadrightarrow_\beta N$. One has in general $\vdash_{\lambda\to} N : B{\to}A{\to}A$, $\not\vdash_{\lambda\to} M : B{\to}A{\to}A$, but $\vdash^{\mathcal{T}\Omega} M : B{\to}A{\to}A$. See exercise 14.3.4.

In $\triangledown =$ HL one has φ as top, but we will not work with $\lambda\cap^{\triangledown\varphi}$ but with $\lambda\cap^{\triangledown}$. In general the type assignment system $\lambda\cap^{\mathcal{T}\Omega}$ will be used for the the $\lambda\mathsf{K}$-calculus and $\lambda\cap^{\mathcal{T}}$ for the restricted $\lambda\mathsf{I}$-calculus. In $\nabla =$ EHR the type ν is the top. See Exercise 14.3.24. This system is used together with the type assignment system $\lambda\cap^{\nu}$ in which (Ω universal) is replaced by (ν universal) stating

$$\Gamma \vdash (\lambda x.M) : \nu$$

for arbitrary Γ and M.

14.2.5. Lemma. *Let $\mathcal{T}$ be an ITS. Then*

(i) $\Gamma \vdash^{\mathcal{T}} M : A \;\Rightarrow\; \mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma)$.

(ii) *If in $\mathcal{T}$ one has $\Omega = \Omega{\to}\Omega$, then*

$$\mathrm{FV}(M) \subseteq \mathrm{dom}(\Gamma) \;\Rightarrow\; \Gamma \vdash^{\mathcal{T}} M : \Omega.$$

Proof. (i) By induction on the derivation.

(ii) By induction on M. ■

Admissible rules

14.2.6. Proposition. *Let $\mathcal{T}$ be an ITS. The following rules are admissible*[2] *in the intersection type assignment system $\lambda\cap^{\mathcal{T}(\Omega)}$.*

---

[2]This means that e.g. $\Gamma \vdash M : A \quad x \notin \Gamma \;\Rightarrow\; \Gamma, x{:}B \vdash M : A$ is valid.

(*weakening*)	$\dfrac{\Gamma \vdash M : A \quad x \notin \Gamma}{\Gamma, x{:}B \vdash M : A};$
(*strengthening*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad x \notin FV(M)}{\Gamma \vdash M : A};$
(*cut*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M[x := N]) : A};$
($\leq$-*L*)	$\dfrac{\Gamma, x{:}B \vdash M : A \quad C \leq B}{\Gamma, x{:}C \vdash M : A};$
($\to$L)	$\dfrac{\Gamma, y{:}B \vdash M : A \quad \Gamma \vdash N : C \quad x \notin \Gamma}{\Gamma, x{:}(C \to B) \vdash (M[y := xN]) : A};$
($\cap$L)	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma, x{:}(A \cap C) \vdash M : B}.$

Figure 14.5: Various admissible rules.

PROOF. By induction on the structure of derivations. ■

The proofs in the following chapters will freely use the rules of the above Proposition.

As we remarked earlier, there are various equivalent alternative presentations of intersection type assignment systems. We have chosen a natural deduction presentation, where $\triangledown$-bases are additive. We could have taken, just as well, a sequent style presentation and replace rule ($\to$E) with the three rules ($\to$L), ($\cap$L) and (cut) occuring in Proposition **??**, see Barbanera et al. [1995], Barendregt and Ghilezan [n.d.]. Next to this we could have formulated the rules so that $\triangledown$-bases "multiply". Notice that because of the presence of the type constructor $\cap$, a special notion of *multiplication of* $\triangledown$*-bases* can be given.

14.2.7. DEFINITION (Multiplication of $\triangledown$-bases).

$$\begin{array}{rcl} \Gamma \uplus \Gamma' & = & \{x{:}\, A \cap B \mid x{:}\, A \in \Gamma \text{ and } x{:}\, B \in \Gamma'\} \\ & \cup & \{x{:}\, A \mid x{:}\, A \in \Gamma \text{ and } x \notin \Gamma'\} \\ & \cup & \{x{:}\, B \mid x{:}\, B \in \Gamma' \text{ and } x \notin \Gamma\}. \ \blacksquare \end{array}$$

Accordingly we define:

$$\Gamma \Subset \Gamma' \;\Leftrightarrow\; \exists \Gamma''.\, \Gamma \uplus \Gamma'' = \Gamma'.$$

For example, $\{x{:}A, y{:}B\} \uplus \{x{:}C, z{:}D\} = \{x{:}A \cap C, y{:}B, z{:}D\}$.

14.2.8. PROPOSITION. *The following rules are admissible in all* $\lambda\cap^{\mathcal{T}(\Omega)}$.

(*multiple weakening*)	$\dfrac{\Gamma_1 \vdash M : A}{\Gamma_1 \uplus \Gamma_2 \vdash M : A}$
(*relevant* $\to$E)	$\dfrac{\Gamma_1 \vdash M : A \to B \quad \Gamma_2 \vdash N : A}{\Gamma_1 \uplus \Gamma_2 \vdash MN : B}$
(*relevant* $\cap$I)	$\dfrac{\Gamma_1 \vdash M : A \quad \Gamma_2 \vdash M : B}{\Gamma_1 \uplus \Gamma_2 \vdash M : A \cap B}$

PROOF. By induction on derivations. ∎

In Exercise 15.4.6, it will be shown that we can replace rule ($\leq_\nabla$) with other more perspicuous rules. This is possible as soon as we will have proved appropriate "inversion" theorems for $\lambda\cap^\nabla$. For some very special theories, one can even do away altogether with rule ($\leq_\nabla$), provided the remaining rules are reformulated "multiplicatively" with respect to ∇-bases, see e.g. Di Gianantonio and Honsell [1993]. We shall not follow up this line of investigation.

In $\lambda\cap$, assumptions are allowed to appear in the basis without any restriction. Alternatively, we might introduce a *relevant* intersection type assignment system, where only "minimal-base" judgments are derivable, (see Honsell and Ronchi Della Rocca [1992]). Rules like (*relevant* $\to$E) and (*relevant* $\cap$I), which exploit the above notion of multiplication of bases, are essential for this purpose. Relevant systems are necessary, for example, for giving finitary logical descriptions of qualitative domains as defined in Girard et al. [1989]. We will also not follow up this line of research. (See Honsell and Ronchi Della Rocca [1992]).

14.2.9. REMARK. For non-extensional intersection type theories, including historically the first intersection type theory CD, a slightly different notion of type assignment system is used. This is treated in exercise 14.3.3.

14.3. Exercises 26.2.2004:1018

14.3.1. Let $\nabla = \mathrm{HL}$, $\leq = \leq_{\mathrm{HL}}$, $\sim = \sim_{\mathrm{HL}}$ and $\mathbb{T} = \mathbb{T}^{\mathrm{HL}}$. Then

(i) $\omega \leq A \leq \varphi$, for all $A \in \mathbb{T}$.

(ii) $$\begin{aligned} A &\leq \varphi\to(\varphi\to(\ldots\to(\varphi\to\omega)..)\cap(\textstyle\bigcap_i C_i) \Rightarrow \\ A &\equiv \varphi\to(\varphi\to(\ldots\to(\varphi\to\omega)..)\cap(\textstyle\bigcap_j D_j), \end{aligned}$$

for some $\vec{D}$. The number of arrows at the RHS may differ from that at the LHS.

(iii) Show that $\varphi \not\leq \omega$.

(iv) Show that each element $A \in \mathbb{T}$ is of the form (up to the order of the elements)

$$A \equiv A_1 \cap \ldots \cap A_n \cap (B_1\to C_1) \cap \ldots \cap (B_n\to C_n),$$

with the $A_i \in \{\omega, \varphi\}$, $B_j, C_j \in \mathbb{T}$. Replacing each $A_i \equiv \omega$ by $\varphi \to \omega$ and $A_i \equiv \varphi$ by $\omega \to \varphi$ we get for some $P_i, Q_i \in \mathbb{T}$, $k > 0$

$$A \sim (P_1 \to Q_1) \cap \ldots \cap (P_k \to Q_k).$$

This RHS is called the *canonical form* of A.

(v) Show that if $A \leq_{\mathrm{HL}} A'$, then for all $1 \leq j \leq k'$ there exists $1 \leq i \leq k$ such that $P'_j \leq P_i$, where $\bigcap_{1 \leq i \leq k}(P_i \to Q_i) \sim A$ and $\bigcap_{1 \leq i \leq k'}(P'_i \to Q'_i) \sim A'$ are the canonical forms.

(vi) Show that $\varphi \not\leq (\varphi \to \varphi)$.

14.3.2. Let again $\nabla = \mathrm{HL}$. Show that if $A \leq A'$, then for all $1 \leq j \leq k'$ there exists $1 \leq i_1, \ldots, i_n \leq k$, $n > 0$, such that $P'_j \leq P_{i_1} \cap \ldots P_{i_n}$ and $Q_{i_1} \cap \ldots Q_{i_n} \leq Q'_j$, where $\bigcap_{1 \leq i \leq k}(P_i \to Q_i) \sim A$ and $\bigcap_{1 \leq i \leq k'}(P'_i \to Q'_i) \sim A'$ are the canonical forms defined in 14.3.1(iv).

14.3.3. Let ∇ be an intersection type theory over $\mathbb{T} = \mathbb{T}(\mathbb{C})$.

(i) A ∇-*statement* is of the form $M : A$ with the *subject* $M \in \Lambda$ an untyped lambda term and the *predicate* $A \in \mathbb{T}$.

(ii) A *declaration* is a ∇-statement of the form $x : A$.

(iii) A ∇-*basis* Γ is is a set of ∇-declarations, with all variables distinct.

(iv) A ∇-*assertion* is of the form $\Gamma \vdash M : A$, where $M : A$ is a ∇-statement and Γ is a ∇-basis.

(v) The *basic intersection type assignment system* derives ∇-assertions $\Gamma \vdash M : A$ by the following axioms and rules.

(Ax)	$\Gamma \vdash x{:}A$	if $(x{:}A) \in \Gamma$
$(\to\mathrm{I})$	$\dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$	
$(\to\mathrm{E})$	$\dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$	
$(\leq)$	$\dfrac{\Gamma \vdash M : A \quad A \leq_\nabla B}{\Gamma \vdash M : B}$	
$(\cap\mathrm{I})$	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$	

We write $\Gamma \vdash^\nabla_B M : A$ if $\Gamma \vdash M : A$ is derivable in this theory. If there is a largest type Ω, then one may add the rule $\Gamma \vdash M : \Omega$ and obtian the system $\vdash^{\nabla\Omega}_B$.

(vi) Let ∇ be an extensional ITT. Show that (cf. the notation of 14.2.3)

$$\Gamma \vdash^{\nabla(\Omega)}_B M : A \Leftrightarrow \Gamma \vdash^{\nabla(\Omega)} M : A.$$

14.3.4. Prove the statements in example 14.2.4.

14.3.5. Let ∇ be an extensional ITT over the constants $\mathbb{C}$.

(i) Show that for $A \in \mathbb{T}_\cap(\mathbb{C})$ one has

$$[\![A]\!]^{\mathcal{T}^\nabla} = [A]_{\sim_\nabla}.$$

(ii) Show that

$$\mathcal{T}^{\triangledown} \models A \leq B \;\Leftrightarrow\; (A \leq B) \in \triangledown\,.$$

14.3.6. Let Γ be the set of first order axioms in the language L of ITSs (one binary relation $\leq$ and two binary operations $\cap$, $\to$).

$$\boxed{\begin{array}{rl} \forall a. & a \leq a \\ \forall a,b,c. & a \leq b \;\&\; b \leq c \;\Rightarrow\; a \leq c \\ \forall a,b. & a \cap b \leq a \\ \forall a,b. & a \cap b \leq b \\ \forall a,b,c. & c \leq a \;\&\; c \leq b \;\Rightarrow\; c \leq (a \cap b) \end{array}}$$

Let $\triangledown$ be an extensional ITT over $\mathbb{T}_{\cap}(\mathbb{C})$. Show that

$$\Gamma, \triangledown \vdash A \leq B \;\Rightarrow\; (A \leq B) \in \triangledown,$$

where $\vdash$ denotes first order provability with language $L(\mathbb{C})$. [Hint. Use $\mathcal{T}^{\triangledown}$.] Show that one may drop the assumption that $\triangledown$ is extensional.

14.3.7. Define the system $\mathcal{D}$, see Krivine [1990], as the type assignment system (with types in $\mathbb{T}^{\mathrm{CD}}$) axiomatised by the axiom (Ax) and the rules ($\to$I), ($\to$E), ($\cap$I), ($\cap$E).

$$\boxed{\begin{array}{llll} & \text{(Ax)} \quad \Gamma \vdash x{:}A \quad \text{if } (x{:}A \in \Gamma) & & \\[6pt] (\to\mathrm{I}) & \dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} & (\to\mathrm{E}) & \dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \\[10pt] (\cap\mathrm{I}) & \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B} & (\cap\mathrm{E}) & \dfrac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \dfrac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B} \end{array}}$$

(i) Show that $\lambda x.xxx$ and $(\lambda x.xxx)\mathsf{S}$ are typable in system $\mathcal{D}$.
(ii) Show that all terms in normal forms are typable in system $\mathcal{D}$.

14.3.8. Show that derivability in system $\mathcal{D}$ as defined in previous exercise coincides with derivability in $\lambda\cap^{\mathrm{CD}}_{\mathcal{B}}$, i.e.

$$\Gamma \vdash^{\mathcal{D}} M : A \;\Leftrightarrow\; \Gamma \vdash^{\mathrm{CD}}_{\mathcal{B}} M : A.$$

14.3.9. The system $\mathcal{D}\Omega$ obtained from $\mathcal{D}$ by allowing types in $\mathbb{T}^{\mathrm{BCD}}$ and adding axiom (Ax-Ω) is studied in Krivine [1990]. Prove that

$$\Gamma \vdash^{\mathcal{D}\Omega} M : A \;\Rightarrow\; \Gamma \vdash^{\mathrm{BCD}\Omega} M : A,$$

but the converse is not valid.

14.3.10. Show the following.
(i) $\vdash^{\mathcal{D}} \lambda z.\mathsf{KI}(zz) : (A \to B) \cap A \to C \to C$.
(ii) $\vdash^{\mathcal{D}\Omega} \lambda z.\mathsf{KI}(zz) : \Omega \to C \to C$.
(iii) $\vdash^{\mathrm{BCD}\Omega} \lambda z.\mathsf{KI}(zz) : B \to C \to C$.

14.3.11. If $\mathcal{T}$ is just a type structure (no intersection), rule $\cap$I will be left out of definition 14.2.2 (i), (ii) and the assignment system is denoted by $\lambda^{\mathcal{T}(\Omega)}$. Are there interesting properties beyond those found in Part II?

14.3.12. Show that all closed terms have type ω in $\lambda\cap^{\text{Park}}_{\Omega}$.

14.3.13. Show that for each number $n\in\mathbb{N}$ there is a type $A_n\in\mathbb{T}^{\text{CD}}$ such that for the Church numerals $\mathbf{c}_n$ one has $\Gamma\vdash^{\mathcal{D}}\mathbf{c}_{n+1}:A_n$ but $\Gamma\nvdash^{\mathcal{D}}\mathbf{c}_n:A_n$.

14.3.14. Derive $\vdash^{\mathcal{CDZ}}(\lambda x.xxx)\mathsf{S}:\varphi$ and $y{:}\omega,z{:}\omega\vdash^{\mathcal{CDZ}}(\lambda x.xxx)(\mathsf{S}yz):\omega$.

14.3.15. Prove that $\vdash^{\mathcal{S}c}\mathsf{J}:A{\to}A$ is derivable for all $A\in\mathbb{T}^{\mathcal{S}c}$.

14.3.16. Find in $\lambda\cap^{\text{BCD}}$ a type for the paradoxical combinator Y (i.e. find an $A\in\mathbb{T}^{\text{BCD}}$ such that on has $\vdash^{\text{BCD}}\mathsf{Y}:A$).

14.3.17. (i) Prove that for all types $A\in\mathbb{T}^{\text{AO}}$ there is an n such that

$$\Omega^n{\to}\Omega\leq_{\text{AO}}A.$$

(ii) Define $\mathsf{K}^\infty\equiv\mathsf{YK}$. This term is called the "ogre". Find a type for it in the system $\lambda\cap^{\text{AO}}$.

(iii) Is there a type which cannot be derived for the "ogre" in $\lambda\cap^{\text{AO}}$? [Hint: use (i) and (ii).]

14.3.18. Find the relation between the following three types in w.r.t. $\leq_{\text{CDZ}}$.

$$(\omega{\to}(\varphi{\to}\varphi){\to}\omega)\cap((\varphi{\to}\varphi){\to}\varphi),(\omega{\to}\omega){\to}\omega\text{ and }\varphi{\to}(\omega{\to}\omega){\to}\varphi.$$

14.3.19. Take the powerset $\mathcal{P}(\mathbb{T}^{\text{CD}})$ and make it an applicative domain by defining $X\cdot Y=\{A\mid\exists B{\in}Y\ B{\to}A{\in}X\}$. Show that this is a model of the λ-I-calculus, by setting

$$[\![M]\!]^{\text{CD}}_\rho=\{A{\in}\mathbb{T}^{\text{CD}}\mid\exists\Gamma\models\rho.\,\Gamma\vdash^{\text{CD}}_{\mathcal{B}}M:A\},$$

where $\Gamma\models\rho\ \Leftrightarrow\ \forall(x{:}B){\in}\Gamma.\ B{\in}\rho(x)$. Does one obtain in this way a model of the full λ-calculus?

14.3.20. Take the powerset $\mathcal{P}(\mathbb{T}^{\text{BCD}})$ and make it an applicative domain as in previous exercise. Show that it induces a model of the full λ-calculus.

14.3.21. Let $A,B{\in}\mathbb{T}^{\text{BCD}}$. Consider the set

$$X=\{C{\to}D\mid A{\to}B\leq_{\text{BCD}}C{\to}D\}.$$

What is the functional behaviour of X with respect to application like in exercise 14.3.19, i.e. which function is represented by $\boldsymbol{\lambda}Y.X\cdot Y$? [Hint. Calculate $X\cdot\{C\mid A\leq C\}$.]

14.3.22. Let ∇_1 and ∇_2 be type theories with $\mathbb{T}^{\nabla_1}\subseteq\mathbb{T}^{\nabla_2}$ (possibly modulo a renaming of type constants). Define $\nabla_1\subseteq\nabla_2$ as

$$\forall\Gamma,M,A.\,[\Gamma\vdash^{\nabla_1}M:A\ \Rightarrow\ \Gamma\vdash^{\nabla_2}M:A].$$

Check that the inclusions of Figure 14.6 are correct. Do you grasp the meaning of the horizontal line in the same figure?

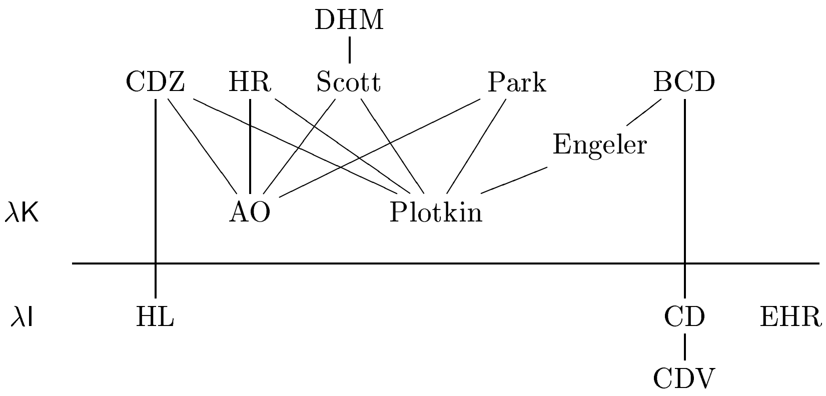

Figure 14.6: Inclusion among some intersection type theories.

14.3.23. Show that in the intersection type theory $\triangledown$ with $\mathbb{C}^{\triangledown} = \{\Omega, \omega\}$ and $\triangledown = \text{Scott} \cup \text{Park}$ one has $\forall A, B{\in}\mathbb{T}.\ A \sim B$.

14.3.24. Let $\triangledown = \text{EHR}$ be the intersection type theory over $\mathbb{T} = \mathbb{T}(\{\nu\})$, axiomatized by the rule (η) and the axiom (ν).

(i) Show that $\forall A{\in}\mathbb{T}\ A \leq_{\text{EHR}} \nu$.

(ii) We define the intersection type assignment system $\lambda\cap^{\nu} = \lambda\cap^{\text{EHR}}$ by the following axioms and rules.

$$
\begin{array}{lll}
(\mathrm{Ax}) & \Gamma \vdash x{:}A & \text{if } (x{:}A){\in}\Gamma \\[2ex]
(\to\mathrm{I}) & \dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A{\to}B} & \\[2ex]
(\to\mathrm{E}) & \dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} & \\[2ex]
(\leq) & \dfrac{\Gamma \vdash M : A \quad A \leq_{\triangledown} B}{\Gamma \vdash M : B} & \\[2ex]
(\cap\mathrm{I}) & \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B} & \\[2ex]
(\nu) & \Gamma \vdash (\lambda x.M) : \nu &
\end{array}
$$

We write $\Gamma \vdash^{\nu} M : A$ if $\Gamma \vdash M : A$ is derivable in this assignment system.

(iii) Question to be asked??

Chapter 15

Basic properties

15.1. Generation Theorems 26.2.2004:1018

We derive generation properties, stating necessary and sufficient conditions when an assertion $\Gamma \vdash^{\mathcal{T}(\Omega)} M : A$ can be derived depending on the form of M and A.

Intersection type theories and structures can be rather wild structures. Throughout this Part, in the style of Coppo et al. [1984] we shall isolate special properties which allow to "reverse" some of the rules of the type assignment system $\vdash^{\mathcal{T}}$, so as to achieve some form of Generation (or Inversion) Theorems (see Theorems 15.1.2 and 15.1.16).

We start by a simple example which shows that in general rule $(\to\mathrm{E})$ cannot be reversed always, i.e. that if $\Gamma \vdash MN : B$, then it is not always true that there exists A such that $\Gamma \vdash M : A{\to}B$ and $\Gamma \vdash N : A$.

15.1.1. Example. Let $\mathcal{T} = \mathcal{T}^{\bigtriangledown}$ with $\bigtriangledown$ = Engeler, one of the intersection type theories. Let $\Gamma = \{x{:}(\varphi_0{\to}\varphi_1) \cap (\varphi_2{\to}\varphi_3), y{:}(\varphi_0 \cap \varphi_2)\}$. Then one has $\Gamma \vdash^{\mathcal{T}} xy : \varphi_1 \cap \varphi_3$. Nevertheless, it is not possible find a type C such that $\Gamma \vdash^{\mathcal{T}} x : C{\to}(\varphi_1 \cap \varphi_3)$ and $\Gamma \vdash^{\mathcal{T}} y : C$.

15.1.2. Theorem (Generation Theorem I). *Let $\mathcal{T}$ be an intersection type structure. Then in $\lambda\cap^{\mathcal{T}}$ the following hold unconditionally and in $\lambda\cap^{\mathcal{T}\Omega}$ provided that in (i), (ii) one has $A \neq \Omega$.*

$$\begin{array}{rrcl}
\text{(i)} & \Gamma \vdash x : A & \Leftrightarrow & \exists B{\in}\mathcal{T}[B \leq A \;\&\; (x{:}B){\in}\Gamma]. \\
\text{(ii)} & \Gamma \vdash MN : A & \Leftrightarrow & \exists k{>}0\, \exists B_1,\dots,B_k,C_1,\dots,C_k{\in}\mathcal{T} \\
& & & [C_1 \cap \dots \cap C_k \leq A \;\&\; \forall i{\in}\{1,\dots,k\} \\
& & & \Gamma \vdash M : B_i{\to}C_i \;\&\; \Gamma \vdash N : B_i]. \\
\text{(iii)} & \Gamma \vdash \lambda x.M : A & \Leftrightarrow & \exists k{>}0\, \exists B_1,\dots,B_k,C_1,\dots,C_k{\in}\mathcal{T} \\
& & & [(B_1{\to}C_1) \cap \dots \cap (B_k{\to}C_k) \leq A \\
& & & \;\&\; \forall i{\in}\{1,\dots,k\}.\Gamma, x{:}B_i \vdash M : C_i].
\end{array}$$

Proof. We only treat $(\Rightarrow)$, as $(\Leftarrow)$ is trivial. First we do this for $\lambda\cap^{\mathcal{T}}$.

(i) By induction on derivations. We reason according which axiom or rule has been used in the last step. Only axiom (Ax), and rules $(\cap\mathrm{I})$, $(\leq)$ can have been applied. In the first case one can take $B = A$. In the other two cases the induction hypothesis applies.

(ii) By induction on derivations. By assumption on A and the shape of the term the last applied step has to be rule ($\to$E), ($\leq$) or ($\cap$I). In the first case the last applied rule is

$$(\to\text{E})\quad \frac{\Gamma \vdash M : D{\to}A \quad \Gamma \vdash N : D}{\Gamma \vdash MN : A}.$$

We can take $k = 1$ and $C_1 = A$ and $B_1 = D$. In the second case the last rule applied is

$$(\leq)\quad \frac{\Gamma \vdash MN : B \quad B \leq A}{\Gamma \vdash MN : A}$$

and the induction hypothesis applies. In the last case $A \equiv A_1 \cap A_2$ and the last applied rule is

$$(\cap\text{I})\quad \frac{\Gamma \vdash MN : A_1 \quad \Gamma \vdash MN : A_2}{\Gamma \vdash MN : A_1 \cap A_2}.$$

By the IH there are B_i, C_i, D_j, E_j, with $1 \leq i \leq k,\ 1 \leq j \leq k'$, such that

$$\begin{array}{ll} \Gamma \vdash M : B_i{\to}C_i & \Gamma \vdash N : B_i, \text{ for all } i, \\ \Gamma \vdash M : D_j{\to}E_j & \Gamma \vdash N : D_j, \text{ for all } j, \\ \bigcap_i C_i \leq A_1, & \bigcap_j E_j \leq A_2. \end{array}$$

Hence we are done, as $C_1 \cap \ldots \cap C_k \cap E_1 \cap \ldots \cap E_{k'} \leq A$.

(iii) Again by induction on derivations. We only treat the case $A \equiv A_1 \cap A_2$ and the last applied rule is ($\cap$I):

$$(\cap\text{I})\quad \frac{\Gamma \vdash \lambda x.M : A_1 \quad \Gamma \vdash \lambda x.M : A_2}{\Gamma \vdash \lambda x.M : A_1 \cap A_2}.$$

By the IH there are B_i, C_i, D_j, E_j, with $1 \leq i \leq k,\ 1 \leq j \leq k'$, such that

$$\begin{array}{ll} \forall i[\Gamma, x{:}B_i \vdash M : C_i], & \bigcap_i (B_i{\to}C_i) \leq A_1, \\ \forall j\ [\Gamma, x{:}D_j \vdash M : E_j], & \bigcap_j (D_j{\to}E_j) \leq A_2). \end{array}$$

We are done, since $(B_1{\to}C_1)\cap\ldots\cap(B_k{\to}C_k)\cap(D_1{\to}E_1)\cap\ldots\cap(D_{k'}{\to}E_{k'}) \leq A$.

Now we prove ($\Rightarrow$) in (i), (ii) for $\lambda\cap^{\mathcal{T}\Omega}$.

(i) The condition $A \neq \Omega$ implies that axiom (Ω universal) cannot have been used. Hence the reasoning above suffices.

(ii) The only interesting rule is ($\cap$I). Condition $A \neq \Omega$ implies that we cannot have $A_1 = A_2 = \Omega$. In case $A_1 \neq \Omega$ and $A_2 \neq \Omega$ the result follows as above. The other cases are more easy. ∎

Notice that as a consequence of this Theorem the *subformula property* holds for all $\lambda\cap^{\mathcal{T}}$.

15.1.3. Corollary (Subformula property). *Assume $\Gamma \vdash^{\mathcal{T}} M : A$ and $N \subseteq M$. Then N is typable in an extension $\Gamma^+ = \Gamma, x_1{:}B_1, \ldots, x_n{:}B_n$ in which also the variables $\{\vec{x}\} = \mathrm{FV}(N) - \mathrm{FV}(M)$ get a type assigned.*

PROOF. If N is a subterm of M, then we can write $M \equiv C[N]$. The statement is proved by induction on the structure of $C[\ ]$. ■

15.1.4. PROPOSITION. *Let $\mathcal{T}$ be an ITS. Suppose that $x \notin \mathrm{dom}(\Gamma)$. Write $\vdash$ for $\vdash^{\mathcal{T}(\Omega)}$, i.e. for both cases. Then*

$$\exists B\,[\Gamma \vdash N : B \;\&\; \Gamma \vdash M[x{:}=N] : A] \;\Rightarrow\; \exists B\,[\Gamma \vdash N : B \;\&\; \Gamma, x{:}B \vdash M : A].$$

PROOF. By induction on the structure of M. ■

Under some conditions (that will hold for many ITSs, notably the syntactic ones coming from the ITTs introduced in section 14.1, the generation theorem can be restated in a more memorable form. This will be Theorem 15.1.16.

In case there is a top element Ω, an intersection may be taken over an empty sequence in which case the outcome is Ω.

15.1.5. DEFINITION. Let $\mathcal{T}$ be an ITS.

(i) $\mathcal{T}$ is called *β-sound* iff $\forall k{\geq}1 \forall A_1,\dots,A_k,B_1,\dots,B_k,C,D{\in}\mathcal{T}$

$$\left.\begin{array}{l}(A_1{\to}B_1)\cap\ldots\cap(A_k{\to}B_k) \leq (C{\to}D) \;\Rightarrow\\ C \leq A_{i_1}\cap\ldots\cap A_{i_p} \;\&\; B_{i_1}\cap\ldots\cap B_{i_p} \leq D,\\ \text{for some } p \geq 0 \text{ and } 1 \leq i_1,\dots,i_p \leq k.\end{array}\right\} \qquad (*)$$

(ii) $\mathcal{T}$ is called *sharply β-sound* if in $(*)$ condition $p \geq 0$ is replaced by $p > 0$.

15.1.6. PROPOSITION. *Let $\mathcal{T}$ be a β-sound ITS having a top Ω. Then $(*)$ in definition 15.1.5 also holds if $k = 0$, i.e.*

$$\Omega = C{\to}D \;\Rightarrow\; \Omega = D.$$

PROOF. Suppose $\Omega = C{\to}D$. Then $\Omega{\to}\Omega \leq \Omega = C{\to}D$, hence by β-soundness $C \leq \Omega$ and $\Omega \leq D$, i.e. $\Omega = D$. ■

15.1.7. LEMMA. *Let $\mathcal{T}$ be an ITS with element Ω satisfying (Ω), $(\Omega\eta)$ and (η). Then $A{\to}\Omega = \Omega$, for all $A{\in}\mathcal{T}$.*

PROOF. $A{\to}\Omega \leq_{(\Omega)} \Omega \leq_{(\Omega\eta)} \Omega{\to}\Omega \leq_{(\eta)} A{\to}\Omega$, hence $A{\to}\Omega = \Omega$. ■

15.1.8. COROLLARY. *Let $\mathcal{T}$ be β-sound and satisfy (Ω), $(\Omega\eta)$ and (η) for some $\Omega{\in}\mathcal{T}$. Then for all $A,B{\in}\mathcal{T}$*

$$A{\to}B = \Omega \;\Leftrightarrow\; B = \Omega.$$

PROOF. $(\Rightarrow)$ By Proposition 15.1.6. $(\Leftarrow)$ By Lemma 15.1.7.

Note that if $\mathcal{T}$ is β-sound, then we have $A{\to}B \leq A'{\to}B' \;\Rightarrow\; A' \leq A \;\&\; B \leq B'$ if B' is not the top element (but not in general).

15.1.9. DEFINITION. Let ∇ be an intersection type theory over $\mathbb{T}_\cap(\mathbb{C})$.

(i) A *simple type* (s-type) is an $A \in \mathbb{C}$, not being the postulated Ω. (The φ of HL is an s-type, as it is only an *a posteriori* top.)

(ii) An *arrow type* (a-type) is a type of the form $A \rightarrow B$.

(iii) An *arrow intersection type* (ai-type) is a (possibly empty) intersection of a-types.

Note that in an intersection type theory every type A can be written uniquely, apart from the order, as

$$A = A_1 \cap \ldots \cap A_n \cap (P_1 \rightarrow Q_1) \cap \ldots \cap (P_k \rightarrow Q_k),$$

an intersection of constants ($A_i \in \mathbb{C}$) and a-types. We say that $A \sim B$ is an axiom of $\bigtriangledown$ iff both $A \leq B$ and $B \leq A$ are axioms. The following two properties are both a strengthening of being β-sound, needed as an induction loading to show that our principal interesection type theories have this property.

15.1.10. Definition. Let $\bigtriangledown$ be an intersection type theory.

(i) $\bigtriangledown$ is said to be *strongly* β iff

$$\begin{array}{ll} A_1 \cap \ldots \cap A_n \cap (P_1 \rightarrow Q_1) \cap \ldots \cap (P_k \rightarrow Q_k) & \leq_{\bigtriangledown} \\ A'_1 \cap \ldots \cap A'_{n'} \cap (P'_1 \rightarrow Q'_1) \cap \ldots \cap (P'_{k'} \rightarrow Q'_{k'}) & \end{array}$$

for $n, n', k, k' \geq 0$, with the $\vec{A}, \vec{A}'$ constants, implies that for all $1 \leq j \leq k'$ there exists a $p \geq 0$ and $1 \leq i_1, \ldots, i_p \leq k$ such that

$$[P'_j \leq_{\bigtriangledown} P_{i_1} \cap \ldots \cap P_{i_p} \;\&\; Q_{i_1} \cap \ldots \cap Q_{i_p} \leq_{\bigtriangledown} Q'_j].$$

(ii) $\bigtriangledown$ is said to be *special* if the following hold.

(1) The rules of $\bigtriangledown$ are (trans), (glb) and (η).

(2) Each axiom of $\bigtriangledown$ is either (refl), (inc$_L$), (inc$_R$), (Ω), ($\Omega\eta$), ($\rightarrow\cap$) or of one of the following forms.

$$\begin{array}{ll} A \sim C & \text{with } A \text{ an s-type and } C \text{ an ai-type;} \\ A \leq A' & \text{with } A, A' \text{ s-types.} \end{array}$$

(3) For every s-type A there exists exactly one ai-type A' such that $A \sim A'$ is an axiom.

From this it follows that *every* type A has a *canonical form* $A \sim A'$, with A' an ai-type, obtained by writing $A \equiv B_1 \cap \ldots \cap B_n$, with B_i either a constant or an a-type, and replacing $B_i = \Omega$ (the postulated top) by the empty intersection of a-types and the s-types B_i by the unique ai-type B'_i such that $B_i \sim B'_i$ is an axiom. Note that the canonical form of φ in HL is $\omega \rightarrow \varphi$.

(4) If $A \leq A'$, with A, A' s-types, is an axiom, then for the canonical forms $(P_1 \rightarrow Q_1) \cap \ldots \cap (P_k \rightarrow Q_k) \sim A$ and $(P'_1 \rightarrow Q'_1) \cap \ldots \cap (P'_{k'} \rightarrow Q'_{k'}) \sim A'$ one has that for all $1 \leq j \leq k'$ there exists $1 \leq i \leq k$ such that $P'_j \leq_{\bigtriangledown} P_i$ and $Q_i \leq_{\bigtriangledown} Q'_j$ (hence by (η) one then has $(P_i \rightarrow Q_i) \leq_{\bigtriangledown} (P'_j \rightarrow Q'_j)$).

15.1.11. LEMMA. *Let $\bigtriangledown$ be an extensional ITT and $\mathcal{T}^{\bigtriangledown}$ its ITS.*
(i) *If $\bigtriangledown$ strongly β, then $\mathcal{T}^{\bigtriangledown}$ is β-sound.*
(ii) *If $\bigtriangledown$ special, then $\mathcal{T}^{\bigtriangledown}$ is β-sound.*

PROOF. (i) In the definition of strongly β, take $n = n' = 0$ and $k' = 1$.
(ii) By induction on provability in $\bigtriangledown$ one can show that

$$A \leq_{\bigtriangledown} A' \quad \Rightarrow \quad \begin{array}{l}\text{for all } 1 \leq j \leq k' \text{ there exists } 1 \leq i_1, \ldots, i_p \leq k, \text{ with } p \geq 0, \\ \text{such that } P'_j \leq_{\bigtriangledown} P_{i_1} \cap \ldots \cap P_{i_p} \text{ and } Q_{i_1} \cap \ldots \cap Q_{i_p} \leq_{\bigtriangledown} Q'_j.\end{array}$$

where $\bigcap_{1 \leq i \leq k} P_i \to Q_i \sim A$ and $\bigcap_{1 \leq j \leq k'} P'_j \to Q'_j \sim A'$ are the canonical forms. Then β-soundness follows. ■

15.1.12. THEOREM. *Let ∇ be an ITT in Fig. 14.2. Then*
(i) *$\bigtriangledown$ is either strongly β or special.*
(ii) *For all $\bigtriangledown$ in Fig. 14.2, except CD, $\bigtriangledown$ is extensional and $\mathcal{T}^{\nabla}$ is β-sound.*

PROOF. (i) By induction on derivability one shows that the type theories CD, CDV, EHR, AO, BCD, Plotkin[1], and Engeler[2] all are strongly β. (Note that if Q'_j is Ω, then one can take $\vec{i}$ empty.) Similarly, the type theories HL, Scott, Park, CDZ, HR, DHM are all special (the axiom $(\omega\varphi)$ is the most involved case to consider).
(ii) Now lemma 15.1.11 applies. ■

We will show in 15.1.13-15.1.15 that for $\nabla \in \{\text{HL}, \text{CDV}, \text{Plotkin}, \text{EHR}\}$ the ITS $\mathcal{T}^{\nabla}$ is sharply β-sound.

15.1.13. PROPOSITION. *$\mathcal{T}^{CDV}$ has no top element.*

PROOF. Let $\psi \in \mathbb{C}^{\text{CDV}}$ and let $A = C_1 \cap \ldots \cap C_n$, where $C_i \equiv \psi$. Then

$$A \leq_{\text{CDV}} B \;\Rightarrow\; B = D_1 \cap \ldots \cap D_m,$$

for some $m > 0, \vec{D}$ with $D_j = \psi$. This follows by induction on the generation of $\leq_{\text{CDV}}$. Now the assertion follows immediately. ■

15.1.14. PROPOSITION. *Let $\nabla \in \{HL, Plotkin, EHR\}$ and $k \geq 1$. Then*

$$(A_1 \to B_1) \cap \ldots \cap (A_k \to B_k) \leq_{\nabla} (C \to D) \;\Rightarrow\; C \leq_{\nabla} A_i, \textit{ for some } 1 \leq i \leq k.$$

PROOF. Case $\nabla = \text{HL}$. This is Exercise 14.3.1(v).
Case $\nabla \in \{\text{Plotkin}, \text{EHR}\}$. By induction on the generation of $\leq_{\nabla}$ one easily shows that

$$\begin{array}{ll} A_1 \cap \ldots \cap A_n \cap (P_1 \to Q_1) \cap \ldots \cap (P_k \to Q_k) & \leq_{\nabla} \\ A'_1 \cap \ldots \cap A'_{n'} \cap (P'_1 \to Q'_1) \cap \ldots \cap (P'_{k'} \to Q'_{k'}) & \end{array}$$

for $n, n', k, k' \geq 0$, with the $\vec{A}, \vec{A'}$ constants, implies that for all $1 \leq j \leq k'$ there exists $1 \leq i \leq k$ such that $P'_j \leq_{\nabla} P_i$. The result then follows immediately. ■

[1]Exercise 15.4.3 gives stronger properties of Plotkin.
[2]Exercise 15.4.4 gives stronger properties of Engeler.

15.1.15. PROPOSITION. (i) *Let* $\nabla \in \{HL, CDV, Plotkin, EHR\}$. *Then* $\mathcal{T}^{\nabla}$ *is sharply* β*-sound.*

(ii) *Let* $\nabla \in \{Scott, Park, CDZ, HR, DHM, BCD, AO, Engeler\}$. *Then* $\mathcal{T}^{\nabla}$ *is not sharply* β*-sound.*

PROOF. (i) Immediate from Theorem 15.1.12(ii) and Propositions 15.1.13 and 15.1.14.

(ii) $A{\to}B \leq_{\nabla} \Omega{\to}\Omega$, but in general $\Omega \not\leq A$. ∎

15.1.16. THEOREM (Generation Theorem II). *Let* $\mathcal{T}$ *be an ITS and let* $\vdash$ *denote* $\vdash^{\mathcal{T}(\Omega)}$. *Then of the following properties (i) holds in general, (ii) provided that* $\mathcal{T}$ *is restricted natural and* A *is not the top in case* $\vdash$ *is* $\vdash^{\mathcal{T}\Omega}$ *and (iii) provided that* $\mathcal{T}$ *is* β*-sound and* C *is not the top in case* $\vdash$ *is* $\vdash^{\mathcal{T}}$.

$$\begin{array}{lrcl}
\text{(i)} & \Gamma, x{:}A \vdash x : B & \Leftrightarrow & A \leq B. \\
\text{(ii)} & \Gamma \vdash (MN) : A & \Leftrightarrow & \exists B{\in}\mathcal{T}\,[\Gamma \vdash M : (B{\to}A)\ \&\ \Gamma \vdash N : B]. \\
\text{(iii)} & \Gamma \vdash (\lambda x.M) : (B{\to}C) & \Leftrightarrow & \Gamma, x{:}B \vdash M : C.
\end{array}$$

PROOF. The proof of each ($\Leftarrow$) is easy. So we only treat ($\Rightarrow$).

(i) If B is not the top, then the conclusion follows from theorem 15.1.2(i). If it is the top, then of course the conclusion holds.

(ii) Suppose $\Gamma \vdash MN : A$. Then by theorem 15.1.2(ii) there are $B_1, \ldots, B_k$, $C_1, \ldots, C_k{\in}\mathcal{T}$, with $k > 0$, such that $\bigcap \vec{C} \leq A$, $\Gamma \vdash M : B_i{\to}C_i$ and $\Gamma \vdash N : B_i$. Hence $\Gamma \vdash N : \bigcap \vec{B}$ and $\Gamma \vdash M : \bigcap(B_i{\to}C_i) \leq \bigcap \vec{B}{\to}\bigcap \vec{C} \leq \bigcap \vec{B}{\to}A$, by lemma 14.1.14. So we can take $B = \bigcap \vec{B}$.

(iii) Suppose $\Gamma \vdash (\lambda x.M) : (B{\to}C)$. Then Theorem 15.1.2(iii) applies and we have for some $k > 0$ and $B_1, \ldots, B_k, C_1, \ldots, C_k$

$$\begin{array}{r}
B_1{\to}C_1 \cap \ldots \cap B_k{\to}C_k \leq B{\to}C \\
\Gamma, x{:}B_i \vdash M : C_i, \text{ for all } i.
\end{array}$$

By β-soundness there are $1 \leq i_1, \ldots, i_p \leq k$ such that

$$\begin{array}{c}
B \leq B_{i_1} \cap \ldots \cap B_{i_p}; \\
C_{i_1} \cap \ldots \cap C_{i_p} \leq C.
\end{array}$$

If $p = 0$, then C is the top and $\vdash$ is $\vdash^{\mathcal{T}\Omega}$, by assumption, so we are done. Else

$$\begin{array}{l}
\Gamma, x{:}B \vdash M : C_{i_j}, \text{ for all } 1 \leq j \leq k, \\
\Gamma, x{:}B \vdash M : C_{i_1} \cap \ldots \cap C_{i_k} \leq C. \ ∎
\end{array}$$

15.1.17. REMARK. (i) In general

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : A \not\Rightarrow \exists B, C.A = (B{\to}C)\ \&\ \Gamma, x{:}B \vdash^{\mathcal{T}} M : C.$$

A counterexample is $\vdash^{\mathrm{BCD}} \mathsf{I} : (D{\to}D) \cap (E{\to}E)$, with D, E atomic.

(ii) Also $\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (A{\to}B) \not\Rightarrow \Gamma, x{:}A \vdash^{\mathcal{T}} M : B$. A counterexample is $y{:}A \vdash^{\mathrm{AO}} \lambda x.xy : \Omega{\to}\Omega$, but $y{:}A, x{:}\Omega \not\vdash^{\mathrm{AO}} xy : \Omega$, see exercise 15.4.2.

15.1.18. COROLLARY. *For the ITTs* $\nabla \in \{Scott, Park, CDZ, HR, DHM, BCD, AO, HL, CDV\}$ *and* $\mathcal{T} = \mathcal{T}^{\nabla}$ *properties (i), (ii) and (iii) of Theorem 15.1.16 hold for* $\vdash^{\mathcal{T}}$*, provided that in (iii) the type* C *is not the top. The properties also hold for* $\vdash^{\mathcal{T}\Omega}$*, provided that in (ii) the type* A *is not the top.*

PROOF. For all $\mathcal{T}$ Theorem 15.1.16 applies. ∎

15.1.19. PROPOSITION. *Let* $\triangledown$ *be one of the extensional ITT's of Fig. 14.2, except AO. Then*

$$\Gamma \vdash^{\nabla} (\lambda x.M) : (B{\to}\Omega) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\nabla} M : \Omega.$$

PROOF. Case $\triangledown \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}, \mathrm{BCD}, \mathrm{Engeler}\}$. By 14.2.5.

Case $\nabla = \mathrm{CDV}$. By Proposition 15.1.13.

Case $\triangledown \in \{\mathrm{HL}, \mathrm{Plotkin}, \mathrm{EHR}\}$. Let $\Gamma \vdash \lambda x.M : B{\to}\Omega$. By theorem 15.1.2(iii) we have for some $k > 0$ and $B_1, \ldots, B_k, C_1, \ldots, C_k$

$$\begin{array}{l}(B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \le (B{\to}\Omega).\\ \Gamma, x : B_i \vdash M : C_i, \text{for all } i.\end{array}$$

By Proposition 15.1.14 $B \le B_i$ for some i. Hence $\Gamma, x{:}B \vdash M : C_i$, so $\Gamma, x{:}B \vdash M : \Omega$. ∎

15.2. Subject reduction and expansion

Various subject reduction and expansion properties are proved, for the classical β and η notions of reduction but also for meaningful restrictions, such as Plotkin's call-by-value reduction.

We now study under which circumstances intersection type assignment is preserved under conversion. There are various notions of conversion worthwhile considering: β expansion and reduction, together with some of their *restrictions*, and η expansion and reduction. The restrictions of β-conversion are obtained by requiring the various redexes to be of a special form: Church's $\beta\mathsf{I}$-redexes, Plotkin's β_v-redexes and finally $\beta\mathsf{N}$-redexes. Let us recall first the definitions of these special β-redexes.

15.2.1. DEFINITION. Let $R \equiv (\lambda x.M)N$ be a β-redex.

(i) R is a $\beta\mathsf{I}$-*redex* if $x \in \mathrm{FV}(M)$ (Curry and Feys [1958]).

(ii) R is a β_v-*redex* if N is a variable or an abstraction (Plotkin [1975]).

(iii) R is a $\beta\mathsf{N}$-*redex* if $x \notin \mathrm{FV}(M)$ and N is either a variable or a closed SN (strongly normalizing) term (Honsell and Lenisa [1999]).

We consider the following rules.

$$(\text{R-}red) \quad \frac{M \to_R N \qquad \Gamma \vdash M : A}{\Gamma \vdash N : A}$$

$$(\text{R-}exp) \quad \frac{M \,{}_R{\leftarrow}\, N \qquad \Gamma \vdash M : A}{\Gamma \vdash N : A}$$

where R is a notion of reduction, notably β- or η-reduction. If one of these rules holds in $\lambda\cap^{\mathcal{T}}$, we write $\lambda\cap^{\mathcal{T}} \models (R-\{\text{exp,red}\})$, respectively. We will characterize those systems for which type assignment is preserved under various notions of reduction and expansion. These properties will be crucial in Chapters 16 and 17, where we will discuss (untyped) λ-models induced by these systems.

β-conversion

We first investigate when $\lambda\cap^{\mathcal{T}(\Omega)} \models (\beta\text{-red})$.

15.2.2. Proposition. *Let $\mathcal{T}$ be an intersection type structure.*

(i) $\lambda\cap^{\mathcal{T}} \models (\beta\textit{-red}) \Leftrightarrow [\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}A) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : A]$.
(ii) $\lambda\cap^{\mathcal{T}\Omega} \models (\beta\textit{-red}) \Leftrightarrow [\Gamma \vdash^{\mathcal{T}\Omega} (\lambda x.M) : (B{\to}A) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}\Omega} M : A]$.

Proof. (i) We write $\vdash$ for $\vdash^{\mathcal{T}}$. ($\Rightarrow$) Assume $\Gamma \vdash \lambda x.M : B{\to}A$, which implies $\Gamma, y{:}B \vdash (\lambda x.M)y : A$ by weakening and rule ($\to$E) for a fresh y. Now rule (β-*red*) gives us $\Gamma, y{:}B \vdash M[x{:=}y] : A$. Hence $\Gamma, x{:}B \vdash M : A$.

($\Leftarrow$) Suppose $\Gamma \vdash (\lambda x.M)N : A$ in order to show $\Gamma \vdash M[x{:=}N] : A$. By Theorem 15.1.2(iii), $\Gamma \vdash \lambda x.M : B_i{\to}C_i$, $\Gamma \vdash N : B_i$ and $C_1 \cap \ldots \cap C_k \leq A$, for some $\vec{B}, \vec{C}$. By assumption $\Gamma, x{:}B_i \vdash M : C_i$. Hence by rule (*cut*), Proposition 14.2.6, one has $\Gamma \vdash M[x{:=}N] : C_i$. Therefore $\Gamma \vdash M[x{:=}N] : A$, using rules ($\cap$I) and ($\leq$).

(ii) Similarly. ■

15.2.3. Corollary. (i) *Let $\mathcal{T}$ be β-sound and satisfy (if there is a top Ω)*

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}\Omega) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : \Omega.$$

Then $\lambda\cap^{\mathcal{T}} \models (\beta\textit{-red})$.

(ii) *Let $\mathcal{T}$ be β-sound with a top Ω. Then $\lambda\cap^{\mathcal{T}\Omega} \models (\beta\textit{-red})$.*

Proof. By the Proposition, using Theorem 15.1.16(iii). ■

15.2.4. Corollary. (i) *Let $\nabla \in \{Scott, Park, CDZ, HR, DHM, BCD, HL, CDV, Engeler, Plotkin, EHR\}$. Then $\lambda\cap^{\nabla} \models (\beta\textit{-red})$.*

(ii) *Let $\nabla \in \{Scott, Park, CDZ, HR, DHM, BCD, AO, HL, Engeler, Plotkin\}$. Then $\lambda\cap^{\nabla\Omega} \models (\beta\textit{-red})$.*

Proof. (i) By corollary 15.2.3 and proposition 15.1.19.

(ii) By the previous Corollary and Theorem 15.1.12(ii). ■

In exercise 15.4.2 it will be shown that $\lambda\cap^{\mathrm{AO}} \not\models \beta$-red.

In Section **??** we will discuss two type theories which are not β-sound, but which nevertheless induce type assignment systems validating (β-red).

β-exp

Now we investigate when $\lambda\cap^{\mathcal{T}(\Omega)} \models (\beta\text{-exp})$. As a warm-up, suppose that $\Gamma \vdash M[x:=N] : A$. Then we would like to conclude that N has a type, as it seems to be a subformula, and therefore $\Gamma \vdash (\lambda x.M)N : A$. There are two problems: N may occur several times in $M[x:=N]$, so that it has (should have) in fact several types. In the system $\lambda\!\to$ this problem causes the failure of rule (β-exp). But in the intersection type theories one has $N : B_1 \cap \ldots \cap B_k$ if $N : B_1, \ldots, N : B_k$. Therefore $(\lambda x.M)N$ has a type if $M[x:=N]$ has one. The second problem arrises if N does not occur at all in $M[x:=N]$, i.e. if the redex is a $\lambda\mathsf{K}$-redex. Following the intersection approach, we would like to give to N the intersection over an empty sequence as type. This makes (β-exp) invalid in $\lambda\cap^{\mathcal{T}}$, but valid in systems $\lambda\cap^{\mathcal{T}\Omega}$.

15.2.5. Proposition. *Let $\mathcal{T}$ be an intersection type structure. Write $\lambda\cap$, $\vdash$ for $\lambda\cap^{\mathcal{T}(\Omega)}$, $\vdash^{\mathcal{T}(\Omega)}$, respectively.*

(i) *Suppose $\Gamma \vdash M[x:=N] : A, x \notin \Gamma$ and $x \notin \mathrm{FV}(N)$. Then*

$$\Gamma \vdash (\lambda x.M)N : A \;\Leftrightarrow\; N \textit{ is typable in context } \Gamma.$$

(ii) $\lambda\cap \models (\beta\text{-}exp) \;\Leftrightarrow\; \forall \Gamma, M, N, A$

$$[\Gamma \vdash M[x:=N] : A \;\Rightarrow\; N \textit{ is typable in context } \Gamma].$$

(iii) $\lambda\cap \models (\beta\mathsf{I}\text{-}exp) \;\Leftrightarrow\; \forall \Gamma, M, N, A$ *with* $x\in\mathrm{FV}(M)$

$$[\Gamma \vdash M[x:=N] : A \;\Rightarrow\; N \textit{ is typable in context } \Gamma].$$

Proof. (i) ($\Rightarrow$) By theorem 15.1.2(ii).

($\Leftarrow$) Let $\Gamma \vdash M[x:=N] : A$ and suppose N is typable in context Γ. By proposition 15.1.4 for some B one has

$$\Gamma \vdash N : B \;\&\; \Gamma, x{:}B \vdash M : A.$$

Then $\Gamma \vdash \lambda x.M : (B\to A)$ and hence $\Gamma \vdash (\lambda x.M)N : A$.

(ii) ($\Rightarrow$) Assume $\Gamma \vdash M[x:=N] : A$. Let y be a fresh variable. Then $\Gamma \vdash M[x: = y][y: = N] : A$. Therefore $\Gamma \vdash (\lambda y.M[x: = y])N : A$, by ($\beta$-exp), hence by (i) we are done. ($\Leftarrow$) Assume $\Gamma \vdash P' : A$, with $P \twoheadrightarrow_\beta P'$. By induction on the generation of $P \twoheadrightarrow_\beta P'$ the conclusion follows from (i) and Theorem 15.1.2.

(iii) Similarly to (ii). ∎

The rather contrived statement given in Theorem 15.2.5(ii) above is immediately met for a type structure having a top element. Moreover, it also follows that for $\beta\mathsf{I}$-redexes, that do not throw away their argument, the expansion rule holds in all $\lambda\cap^{\mathcal{T}}$.

15.2.6. Corollary. (i) *For all $\mathcal{T}$ one has $\lambda\cap^{\mathcal{T}} \models (\beta\mathsf{I}\text{-}exp)$.*

(ii) *Let $\mathcal{T}$ have a top. Then $\lambda\cap^{\mathcal{T}\Omega} \models (\beta\text{-}exp)$.*

Proof. (i) By the subformula property 15.1.3, using lemma 14.2.5.

(ii) Trivial, since every term has type Ω. ■

In exercise 15.4.13 it will be shown that $\lambda\cap^{\nu} \not\models (\beta\mathsf{I}\text{-exp})$.

Other restricted β-expansion results hold universally or are to state.

15.2.7. Definition. Rule ($\beta\mathsf{N}$-exp$'$) is the restriction of ($\beta\mathsf{N}$-exp) to bases Γ assigning types to all variables, i.e. $x{\in}\mathrm{dom}(\Gamma)$ for all $x{\in}\mathsf{V}$.

15.2.8. Corollary. $\lambda\cap^{\mathcal{T}} \models (\beta\mathsf{N}\text{-}exp')$ *for all* $\mathcal{T}$.

Proof. Suppose $P \to_{\beta\mathsf{N}} P'$ by contracting $(\lambda x.M)N$ and $\Gamma \vdash^{\triangledown} P' : A$. Then N is a closed SN term or a variable. In Theorem 17.1.5(iii) it will be proved that each closed SN term is typable from the empty basis. Moreover, by the extra restriction, free variables can all be typed in Γ. Therefore the proof of Theorem 15.2.5 applies. ■

Now we can harvest results towards closure under β-conversion.

15.2.9. Theorem. (i) *Let* $\mathcal{T}$ *be* β*-sound and satisfy (if there is a top* Ω*)*

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}\Omega) \;\Rightarrow\; \Gamma, x{:}B \vdash^{\mathcal{T}} M : \Omega.$$

Then for fresh x *(i.e.* $x \notin \Gamma$*) one has*

$$x{\in}\mathrm{FV}(M) \;\Rightarrow\; [\Gamma \vdash^{\mathcal{T}} (\lambda x.M)N : A \;\Leftrightarrow\; \Gamma \vdash^{\mathcal{T}} M[x{:}=N] : A].$$

(ii) *Let* $\mathcal{T}$ *be* β*-sound and have a top. Then for* $x \notin \Gamma, x \notin \mathrm{FV}(N)$*, one has*

$$\Gamma \vdash^{\mathcal{T}\Omega} (\lambda x.M)N : A \;\Leftrightarrow\; \Gamma \vdash^{\mathcal{T}\Omega} M[x{:}=N] : A.$$

Proof. (i) By Corollaries 15.2.3(i) and 15.2.6(i).

(ii) By Corollaries 15.2.3(ii) and 15.2.6(ii). ■

15.2.10. Corollary. (i) *Let* $\nabla{\in}\{Scott, Park, CDZ, HR, DHM, BCD, HL, CDV, Engeler, Plotkin, EHR\}$*. Then*

$$x{\in}\mathrm{FV}(M) \;\Rightarrow\; [\Gamma \vdash^{\triangledown} (\lambda x.M)N : A \;\Leftrightarrow\; \Gamma \vdash^{\triangledown} M[x{:}=N] : A].$$

(ii) *Let* $\nabla{\in}\{Scott, Park, CDZ, HR, DHM, BCD, AO, HL, Engeler, Plotkin\}$*. Then*

$$\Gamma \vdash^{\triangledown\Omega} (\lambda x.M)N : A \;\Leftrightarrow\; \Gamma \vdash^{\triangledown\Omega} M[x{:}=N] : A.$$

Proof. (i) By Theorem 15.1.12(ii) $\triangledown$ is β-sound. Hence, by Proposition 15.1.19 and Corollary 15.2.3, it follows that $\lambda\cap^{\triangledown} \models (\beta\text{-red})$. Therefore one has
$\Gamma \vdash^{\triangledown} (\lambda x.M)N : A \;\Rightarrow\; \Gamma \vdash^{\triangledown} M[x{:}=N] : A$. The converse follows from $\lambda\cap^{\triangledown} \models (\beta\mathsf{I}\text{-exp})$, Corollary 15.2.6(ii).

(ii) EHR, CDV are left out as they have no top. But as we work with the system in which Ω is the universal type, now Corollary 15.2.6(i) applies. ■

η-conversion

In order to characterize the admissibility of rule (η-*exp*), we need to introduce a further condition on type structures. This condition is necessary and sufficient to derive from the basis $x{:}A$ the same type A for $\lambda y.xy$, as we will show in the proof of Theorem 15.2.12.

15.2.11. Definition. Let $\mathcal{T}$ be an intersection type structure.

(i) $\mathcal{T}$ is called *η-sound* iff for all $A{\in}\mathcal{T}$ there are $k \geq 0$ and $\vec{B} = B_1, \ldots, B_k$, $\vec{C} = C_1, \ldots, C_k, \vec{D}, \vec{E}$, with

$$\begin{array}{l} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \quad \leq \quad A \\ \& \;\; A \;\; \leq \;\; (D_{11}{\to}E_{11}) \cap \ldots \cap (D_{1m_1}{\to}E_{1m_1}) \cap \\ \qquad\qquad \ldots \\ \qquad\qquad (D_{k1}{\to}E_{k1}) \cap \ldots \cap (D_{km_k}{\to}E_{km_k}) \\ \& \;\; B_i \;\leq\; D_{i1} \cap \ldots \cap D_{im_i} \;\&\; E_{i1} \cap \ldots \cap E_{im_i} \;\leq\; C_i, \\ \qquad \text{for } 1 \leq i \leq k. \end{array}$$

(ii) Let $\mathcal{T}$ have a top element Ω. $\mathcal{T}$ is called *$\eta\Omega$-sound* iff for all $A{\in}\mathcal{T}$ at least one of the following two conditions holds.

(1) There are types $\vec{B}$ with $(B_1{\to}\Omega) \cap \ldots \cap (B_n{\to}\Omega) \leq A$;

(2) There are $\vec{B}, \vec{C}, \vec{D}, \vec{E}$, with $\vec{C} = C_1, \ldots, C_k,\ k > 0$, all $\neq \Omega$ and

$$\begin{array}{l} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \cap \\ \cap\, (B_{k+1}{\to}\Omega) \cap \ldots \cap (B_n{\to}\Omega) \quad \leq \quad A \\ \& \;\; A \;\; \leq \;\; (D_{11}{\to}E_{11}) \cap \ldots \cap (D_{1m_1}{\to}E_{1m_1}) \cap \\ \qquad\qquad \ldots \\ \qquad\qquad (D_{k1}{\to}E_{k1}) \cap \ldots \cap (D_{km_k}{\to}E_{km_k}) \\ \& \;\; B_i \;\leq\; D_{i1} \cap \ldots \cap D_{im_i} \;\&\; E_{i1} \cap \ldots \cap E_{im_i} \;\leq\; C_i, \\ \qquad \text{for } 1 \leq i \leq k. \end{array}$$

The validity of η-expansion can be given as follows.

15.2.12. Theorem. *(Characterization of η-exp) Let $\mathcal{T}$ be an ITS.*

(i) $\lambda\cap^{\mathcal{T}} \models (\eta\text{-}exp) \quad \Leftrightarrow \quad \mathcal{T}$ *is η-sound.*

(ii) $\lambda\cap^{\mathcal{T}\Omega} \models (\eta\text{-}exp) \quad \Leftrightarrow \quad \mathcal{T}$ *is $\eta\Omega$-sound.*

Proof. (i) ($\Rightarrow$) Assume $\lambda\cap^{\mathcal{T}} \models (\eta\text{-exp})$ and let $A{\in}\mathcal{T}$, in order to show that it satisfies the condition. As $x{:}A \vdash x : A$, by assumption we have $x{:}A \vdash \lambda y.xy : A$. From Theorem 15.1.2(iii) it follows that $x{:}A, y{:}B_i \vdash xy : C_i$ and $\cap_i(B_i{\to}C_i) \leq A$ for some $B_i, C_i{\in}\mathcal{T}$. By Theorem 15.1.2(ii) for each i there exist $D_{i,j}, E_{i,j}$, such that for each j one has $x{:}A, y{:}B_i \vdash x : D_{i,j}{\to}E_{i,j}$, $x{:}A, y{:}B_i \vdash y : D_{i,j}$ and $\cap_j E_{i,j} \leq C_i$. Hence by Theorem 15.1.2(i) we have $A \leq D_{i,j}{\to}E_{i,j}$ and $B_i \leq D_{i,j}$ for all i and j. Therefore we obtain the condition of 15.2.11(i).

($\Leftarrow$) Suppose that $\Gamma \vdash M : A$ in order to show $\Gamma \vdash \lambda x.Mx : A$, whith x fresh. By assumption A satisfies the condition of 15.2.11(i).

$$\begin{array}{l} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k) \quad \leq \quad A \\ \& \ A \ \leq \ (D_{11}{\to}E_{11}) \cap \ldots \cap (D_{1m_1}{\to}E_{1m_1}) \cap \\ \qquad\qquad \ldots \\ \qquad\qquad (D_{k1}{\to}E_{k1}) \cap \ldots \cap (D_{km_k}{\to}E_{km_k}) \\ \& \quad B_i \ \leq \ D_{i1} \cap \ldots \cap D_{im_i} \ \& \ E_{i1} \cap \ldots \cap E_{im_i} \ \leq \ C_i, \\ \qquad \text{for } 1 \leq i \leq k. \end{array}$$

By rule ($\leq$) for all i,j we have $\Gamma \vdash M : D_{i,j}{\to}E_{i,j}$ and so $\Gamma, x{:}D_{i,j} \vdash Mx : E_{i,j}$ by rule ($\to$E). From ($\leq$ L), ($\cap$I) and ($\leq$) we get $\Gamma, x{:}B_i \vdash Mx : C_i$ and this implies $\Gamma \vdash \lambda x.Mx : B_i{\to}C_i$, using rule ($\to$I). So we can conclude by ($\cap$I) and ($\leq$) that $\Gamma \vdash \lambda x.Mx : A$.

(ii) Now write $\vdash$ for $\vdash^{\mathcal{T}\Omega}$. The proof is nearly the same as for (i). ($\Rightarrow$) Again we get $x{:}A, y{:}B_i \vdash xy : C_i$ and $\cap_i(B_i{\to}C_i) \leq A$ for some $B_i, C_i{\in}\mathcal{T}$. If all $C_i = \Omega$, then A satisfies the first condition of 15.2.11(ii). Otherwise, consider the i such that $C_i \neq \Omega$ and reason as in the proof of ($\Rightarrow$) for (i).

($\Leftarrow$) Suppose that $\Gamma \vdash M : A$ in order to show $\Gamma \vdash \lambda x.Mx : A$, whith x fresh. If A satisfies the first condition of 15.2.11(ii), i.e. $(B_1{\to}\Omega) \cap \ldots \cap (B_n{\to}\Omega) \leq A$, then $\Gamma, x{:}B_i \vdash Mx : \Omega$, hence $\Gamma \vdash \lambda x.Mx : (B_1{\to}\Omega) \cap \ldots \cap (B_n{\to}\Omega) \leq A$. Now let A satisfy the second condition. Then the proof is similar to that for ($\Leftarrow$) in (i). ∎

For most syntactic intersection type structures $\mathcal{T} = \mathcal{T}^{\triangledown}$ of interest the condition of $\eta(\Omega)$ soundness is implied by the following proposition.

15.2.13. PROPOSITION. *Let $\triangledown$ with constants $\mathbb{C}$ satisfy ($\to$-$\cap$) and rule (η).*

(i) *$\mathcal{T}^{\triangledown}$ is η-sound $\Leftrightarrow$*

$$\forall A{\in}\mathbb{C}\, \exists \vec{B}, \vec{C}{\in}\mathbb{T}\, A = (B_1{\to}C_1) \cap \ldots \cap (B_n{\to}C_n), n \geq 0.$$

(ii) *Suppose moreover that $\triangledown$ has a top Ω. Then*

$$\mathcal{T}^{\triangledown} \textit{ is } \eta\Omega\textit{-sound} \Leftrightarrow \forall A{\in}\mathbb{C}\, \exists \vec{B}, \vec{C}{\in}\mathbb{T} \quad \begin{array}{l} A = (B_1{\to}C_1) \cap \ldots \cap (B_n{\to}C_n) \textit{ or} \\ A \geq (B_1{\to}\Omega) \cap \ldots \cap (B_n{\to}\Omega), n \geq 0. \end{array}$$

PROOF. (i) ($\Rightarrow$) Suppose $\mathcal{T}^{\triangledown}$ is η-sound. Let $A{\in}\mathbb{C}$. Then A (modulo $\sim_{\triangledown}$) satisfies the condition of 15.2.11(i), for some $\vec{B}, \vec{C}, \vec{D}, \vec{E}{\in}\mathbb{T}(\mathbb{C})$. By ($\to\cap$) and ($\eta$), using Proposition 14.1.14, it follows that

$$\begin{array}{rcl} A & \leq & (\cap_j D_{1j}{\to} \cap_j E_{1j}) \cap \ldots (\cap_j D_{kj}{\to} \cap_j E_{kj}) \\ & \leq & (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k), \end{array}$$

hence $A {\sim_{\triangledown}} (B_1{\to}C_1) \cap \ldots \cap (B_k{\to}C_k)$.

($\Leftarrow$) By induction on $A{\in}\mathbb{T}(\mathbb{C})$ one can show that $[A]_{\sim_{\triangledown}}$ satisfies the condition of η-soundness. (The case $A_1{\to}A_2$ is trivial.)

(ii) Similarly. ∎

In exercise 15.4.14 it will be proved that under the extra assumption that ∇ satisfies $(\Omega\eta)$ the condition for being $\eta\Omega$-sound can be simplified even more.

15.2.14. COROLLARY. *Let* $\triangledown \in \{Scott, Park, CDZ, HR, DHM, AO, HL\}$. *Then* $\mathcal{T}^{\triangledown}$ *is* η*-sound and* $\eta\Omega$*-sound.*

PROOF. Easy by 15.2.13. ∎

15.2.15. COROLLARY. *Let* $\triangledown \in \{Scott, Park, CDZ, HR, DHM, AO, HL\}$. *Then*

$$\lambda\cap^{\nabla(\Omega)} \models (\eta\text{-}exp).$$

In exercise 15.4.15 the remaining systems are shown not to satisfy (η-exp).

Now we give necessary and sufficient conditions for a a system $\lambda\cap^{\mathcal{T}}$ to satisfy the rule (η-red).

15.2.16. THEOREM. *Consider the intersection type system* $\mathcal{T}$.

(i) *Suppose* $\mathcal{T}$ *has a top* Ω. *Then*

$$\lambda\cap^{\mathcal{T}\Omega} \models (\eta\text{-}red) \;\Leftrightarrow\; \mathcal{T} \textit{ is natural.}$$

(ii) $\lambda\cap^{\mathcal{T}} \models (\eta\text{-}red) \;\Leftrightarrow\; \mathcal{T}$ *is restricted natural.*

PROOF. (i) ($\Rightarrow$) Assume $\lambda\cap^{\mathcal{T}\Omega} \models (\eta\text{-red})$ towards $(\to\cap)$, (η) and $(\Omega\eta)$.

As to $(\to\cap)$, assume $(A\to B)\cap(A\to C) \leq A\to(B\cap C)$. One has

$$x{:}(A\to B)\cap(A\to C), y{:}A \vdash xy : B\cap C,$$

hence by ($\to$I) it follows that $x{:}(A\to B)\cap(A\to C) \vdash \lambda y.xy : A\to(B\cap C)$. Therefore $x{:}(A\to B)\cap(A\to C) \vdash x : A\to(B\cap C)$, by ($\eta$-red). By theorem 15.1.16(i) one can conclude $(A\to B)\cap(A\to C) \leq A\to(B\cap C)$.

As to (η), suppose that $A \leq B$ and $C \leq D$, in order to show $B\to C \leq A\to D$. One has $x{:}B\to C, y{:}A \vdash xy : C \leq D$, so $x{:}B\to C \vdash \lambda y.xy : A\to D$. Therefore by ($\eta$-red) it follows that $x{:}B\to C \vdash x : A\to D$ and we are done as before.

As to $\Omega \leq \Omega\to\Omega$, notice that $x{:}\Omega, y{:}\Omega \vdash xy : \Omega$, so we have $x{:}\Omega \vdash \lambda y.xy : \Omega\to\Omega$. Therefore $x{:}\Omega \vdash x : \Omega\to\Omega$ and again we are done.

($\Leftarrow$) Let $\mathcal{T}$ be natural. Assume that $\Gamma \vdash \lambda x.Mx : A$, with $x \notin \mathrm{FV}(M)$, in order to show $\Gamma \vdash M : A$. If $A = \Omega$, we are done. Otherwise,

$$\begin{aligned}\Gamma \vdash \lambda x.Mx : A \quad \Rightarrow \quad & \Gamma, x{:}B_i \vdash Mx : C_i, 1 \leq i \leq n, \;\& \\ & \bigcap_i (B_i\to C_i) \leq A, \text{ for some } \vec{B}, \vec{C},\end{aligned}$$

by Theorem 15.1.2(iii). By lemma 15.1.8 we omit the i such that $C_i = \Omega$. There is at least one $C_i \neq \Omega$, since otherwise $A \geq \cap_i(B_i\to\Omega) = \Omega$, by lemma 15.1.8, and we would have $A = \Omega$. Hence by theorem 15.1.16(ii)

$$\begin{array}{lll}\Rightarrow & \Gamma, x{:}B_i \vdash M : D_i\to C_i, & \text{for some } \vec{D}, \\ & \Gamma, x{:}B_i \vdash x : D_i & \\ \Rightarrow & B_i \leq D_i, & \text{by theorem 15.1.16(i)}, \\ \Rightarrow & \Gamma \vdash M : B_i\to C_i, & \text{by (strengthening) and } (\eta), \\ \Rightarrow & \Gamma \vdash M : \cap_i(B_i\to C_i) & \\ \Rightarrow & \Gamma \vdash M : A, & \text{since } \cap_i(B_i\to C_i) \leq A.\end{array}$$

(ii) Similarly, but simpler. ■

15.2.17. COROLLARY. (i) *For* $\triangledown \in \{Scott, Park, CDZ, HR, DHM, BCD\}$ *one has* $\lambda\cap^{\triangledown\Omega} \models (\eta\text{-}red)$.

(ii) *Let* $\triangledown \in \{Scott, Park, CDZ, HR, DHM, BCD, AO, HL, CDV\}$. *Then one also has* $\lambda\cap^{\triangledown} \models (\eta\text{-}red)$.

In exercise 15.4.2 it will be shown that $\lambda\cap^{\mathrm{AO}} \not\models (\Omega\eta)$.

Now we can harvest results towards closure under η-conversion. In the following expressions $\lambda x.Mx$ we assume that x is fresh, i.e. $x \notin \mathrm{FV}(M)$.

15.2.18. THEOREM. (i) *Let* $\mathcal{T}$ *be an ITS. Then*

$$\forall \Gamma, M, A[\Gamma \vdash^{\mathcal{T}} M : A \;\Leftrightarrow\; \Gamma \vdash^{\mathcal{T}} \lambda x.Mx : A] \;\Leftrightarrow$$

$$\mathcal{T} \textit{ is restricted natural and } \eta\textit{-sound.}$$

(ii) *Let* $\mathcal{T}$ *moreover have a top* Ω. *Then*

$$\forall \Gamma, M, A[\Gamma \vdash^{\mathcal{T}\Omega} M : A \;\Leftrightarrow\; \Gamma \vdash^{\mathcal{T}\Omega} \lambda x.Mx : A] \;\Leftrightarrow$$

$$\mathcal{T} \textit{ is natural and } \eta\Omega\textit{-sound.}$$

PROOF. (i) By Theorems 15.2.12(i) and 15.2.16(ii).

(ii) By Theorems 15.2.12(ii) and 15.2.16(i). ■

15.2.19. THEOREM. (i) *For* $\nabla \in \{Scott, Park, CDZ, HR, DHM\}$ *one has*

$$\Gamma \vdash^{\nabla\Omega} M : A \;\Leftrightarrow\; \Gamma \vdash^{\nabla\Omega} \lambda x.Mx : A.$$

(ii) *For* $\nabla \in \{Scott, Park, CDZ, HR, DHM, AO, HL\}$ *one has*

$$\Gamma \vdash^{\nabla} M : A \;\Leftrightarrow\; \Gamma \vdash^{\nabla} \lambda x.Mx : A.$$

PROOF. (i) By Corollaries 15.2.17(i) and 15.2.15.

(ii) By Corollaries 15.2.17(ii) and 15.2.15. ■

15.3. Undecidability of inhabitation

In this section we consider type theories with variables, as described in section 14.1. More precisely, we are concerned here with the theories $\mathcal{T}_\diamond^{\eta}$ and $\mathcal{T}_\diamond^{\Omega\eta}$ with type variables. Since we do not consider other type theories here, let us adopt the convention that (from now on) the symbol $\triangledown$ ranges over the set $\{\eta, \Omega\eta\}$.

We investigate the *inhabitation problem* for these type theories, which is to determine, for a given a type τ, if there exists a closed term of type τ (the inhabitant). In symbols, the problem can be presented as follows:

$$\vdash_\diamond^{\triangledown} \; ? : A$$

A slightly more general variant of the problem is the inhabitation problem *relativized* to a given context Γ:

$$\Gamma \vdash_\diamond^{\triangledown} \; ? : A$$

It is however not difficult to show that these two problems are equivalent.

15.3.1. LEMMA.
Let $\Gamma = \{(x_i : A_i) | i = 1, \ldots, n\}$. Then, for all A, the following conditions are equivalent:

1. *There exists a term M such that $\Gamma \vdash^{\nabla}_{\diamond} M : A$;*
2. *There exists a term N such that $\vdash^{\nabla}_{\diamond} N : A_1 \to \cdots \to A_n \to A$.*

PROOF. [3]
If (1) holds, then we can define N as $\lambda x_1 \ldots x_n. M$. If (2) holds, take $M := N x_1 \cdots x_n$. ∎

The main result of the present section (Theorem 15.3.28) is that type inhabitation is undecidable for both $\mathcal{T}^{\eta}_{\diamond}$ and $\mathcal{T}^{\Omega\eta}_{\diamond}$. (Compare this to **??** [4] which states that for simple types the problem is decidable in polynomial space.) Due to Lemma 15.3.1 one only needs to show undecidability of the relativized inhabitation problem. In fact, it is enough to prove undecidability for system $\mathcal{T}^{\eta}_{\diamond}$. Indeed, the result for $\mathcal{T}^{\Omega\eta}_{\diamond}$ is an immediate consequence of the following conservativity lemma:[5]

15.3.2. LEMMA. *If M is in normal form and $\Gamma \vdash^{\Omega\eta}_{\diamond} M : \sigma$, where Γ and σ are ω-free, then also $\Gamma \vdash^{\eta}_{\diamond} M : \sigma$.*

PROOF. *Should we include a proof? Refer to* **??** ∎

Thus, the inhabitation problem for $\mathcal{T}^{\eta}_{\diamond}$ is just a special case of inhabitation in $\mathcal{T}^{\Omega\eta}_{\diamond}$. Should the latter be decidable, the former would have to be decidable too.

The main idea of the undecidability proof is based on the following observation: The process of solving an instance of the inhabitation problem can be seen as a certain (solitary) game of building trees. In this way, one can obtain a combinatorical representation of the computational contents of the inhabitation problem (for a restricted class of types). We call this model a "tree game". It occurs that in order to win a tree-game, the player may be forced to execute a computation of a particular linear bounded automaton (a "typewriter automaton").

Thus, the global strategy of our proof is as follows:

- Show that the emptiness problem for typewriter automata (whether a given automaton accepts at least one word) is undecidable (Lemma 15.3.23 and Exercise 15.4.22).
- Reduce the above to the problem of winning a tree game (Proposition 15.3.27).
- Reduce the tree game problem to (relativized) inhabitation (Corollary 15.3.21).

[3] *Or make it an exercise?*
[4] *reference?*
[5] *This lemma could be mentioned in section 1.1 (with or without proof) rather than here. After all, this is a nice conservativity result.*

Title to be determined

The lemmas of the present subsection (and the next one?) may perhaps be incorporated into section 1.1.

We begin with some basic observations concerning inequalities between types containing type variables.

15.3.3. LEMMA.

1. *Let* $[\bigcap_{i\in I} A_i \leq \alpha]\in\mathcal{T}_\diamond^\eta$, *where* α *is a type variable, and none of the* A_i*'s is an intersection. Then* $\alpha = A_i$, *for some* $i\in I$.

2. *Let* $[\bigcap_{i\in I} \alpha_i \leq A]\in\mathcal{T}_\diamond^\eta$, *where* α_i *are variables and* A *is not an intersection. Then* $A = \alpha_i$, *for some* $i\in I$.

PROOF. Exercise 15.4.18 ∎

We work here with $\mathcal{T}_\diamond^\eta$, and it is perhaps useful to recall here that this theory is applicative, Ω-ν-sound and beta (Proposition **??** 1.1.9). That is, we can use Lemma **??** 1.1.10 (i), (iv), (v) and (vi) and property (3) of **??** 1.1.8. A useful special case of the latter property is as follows:

15.3.4. LEMMA.
If $[\bigcap_{i\in I}(A_i \to \alpha_i) \leq C \to \alpha]\in\mathcal{T}_\diamond^\eta$, *then* $\alpha = \alpha_i$ *and* $[C \leq A_i]\in\mathcal{T}_\diamond^\eta$, *for some* $i\in I$.

PROOF. Immediate. ∎

A consequence of Lemma **??** 1.1.10 (v) and Lemma 15.3.3(1) is:

15.3.5. LEMMA.
If $\Gamma \vdash_\diamond^\eta \lambda x.M : A$ *then* A *is not a variable.*

PROOF. Immediate. ∎

Game contexts

In order to prove that a general decision problem is undecidable, it is enough to identify a "sufficiently difficult" fragment of the problem and prove undecidability of that fragment. Such an approach is often useful. This is because restricting the consideration to specific instances may simplify the analysis of the problem. Of course the choice should be done so that the "core" of the problem remains within the selected special case. This is the strategy we are applying for our inhabitation problem. Namely, we restrict our analysis to the following special case of relativized inhabitation:

$$\Gamma \vdash_\diamond^\eta \, ? : \alpha,$$

where α is a type variable, and Γ is a "game context", the notion of game context being defined as follows:

15.3.6. DEFINITION. We consider types of the following forms:

(A) $\alpha_1 \cap \cdots \cap \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are type variables;

(B) $(\alpha_1 \to \beta_1) \cap \cdots \cap (\alpha_n \to \beta_n)$, where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are type variables;

($\frac{1}{2}$C) $(A_1 \to \alpha_1) \cap (A_2 \to \alpha_2) \to \beta$, where $\alpha_1, \alpha_2, \beta$ are type variables, and A_1, A_2 are of the form (B);

(C) $B_1 \cap \cdots \cap B_n$, where $B_1, \ldots, B_n$ are of the form ($\frac{1}{2}$C).

Types of the forms (A), (B) and (C) are called *game types*. A type context Γ is a *game context* iff all types in Γ are game types.[6]

We now state some technical properties of type judgements involving game types.

15.3.7. LEMMA.
Let Γ be a game context. Then:

1. *If $\Gamma \vdash xM : A$ then A is an intersection of type variables, i.e., $A = \alpha_1 \cap \ldots \cap \alpha_n$, with $n \geq 1$. In particular, it follows that no application of the form xMN is typable in Γ.*

2. *If $\Gamma \vdash xM : \alpha$, where $\Gamma(x) = \bigcap_{i \in I}(B_i \to \alpha_i)$, then $\alpha = \alpha_i$ and $\Gamma \vdash M : B_i$, for some $i \in I$.*[7]

PROOF. To show part (1), suppose $\Gamma \vdash xM : A$. By Lemma **??** 1.1.10 (iv) and (i) we have $[\Gamma(x) \leq B \to A] \in \mathcal{T}^{\eta}_{\diamond}$, for some B, with $\Gamma \vdash M : B$. By Lemma 15.3.3, this cannot happen if $\Gamma(x)$ is of the form (A), and thus $\Gamma(x)$ must be of the form $\bigcap_{i \in I}(B_i \to \alpha_i)$. Since $\mathcal{T}^{\eta}_{\diamond}$ is a beta theory, we have $[\bigcap_{j \in J} \alpha_j \leq A] \in \mathcal{T}^{\eta}_{\diamond}$, for some nonempty subset $J \subseteq I$. Let $A = \bigcap_{k \in K} A_k$, where A_k's are not intersections. Then, by Lemma 15.3.3, we conclude that each A_k equals to some α_i, with $i \in I$. This proves part (1).

For part (2), assume in addition that A is a type variable α. From the above we already have that α must actually be α_i, for some i. By Lemma 15.3.4, the inequality $[\Gamma(x) \leq B \to \alpha] \in \mathcal{T}^{\eta}_{\diamond}$ implies $[B \leq B_i] \in \mathcal{T}^{\eta}_{\diamond}$, for some $i \in I$, from which we deduce $\Gamma \vdash M : B_i$. ■

15.3.8. LEMMA. *If A is a game type and B is of the form $(B_1 \to \alpha_1) \cap (B_2 \to \alpha_2)$, with B_1 and B_2 of the form (B), then $[A \leq B] \not\in \mathcal{T}^{\eta}_{\diamond}$.*

PROOF. Repeated use of Lemmas 15.3.3 and 15.3.4. ■

For game contexts we can state the following refinement of the Generation Lemma **??** 1.1.10, which will be our main tool.

[6]Note that the game types are of rank at most 3 in the sense of ?. *Should we talk about ranks here?*

[7]*The notation $\Gamma(x)$ is very useful. I suggest a global definition.*

15.3.9. LEMMA. *Let Γ be a game context, and let M be in normal form. Then:*

1. *If $\Gamma \vdash M : \alpha$, where α is a type variable, then there are two possibilities:*
   - *Either M is a variable z and $\Gamma(z)$ is of the form (A), where α is one of the components;*
   - *or $M \equiv xP$, where $\Gamma(x) = (B_1 \to \beta_1) \cap \cdots \cap (B_n \to \beta_n)$, and for some i, we have $\alpha = \beta_i$ and $\Gamma \vdash P : B_i$.*
2. *If $\Gamma \vdash M : (A_1 \to \alpha_1) \cap (A_2 \to \alpha_2)$, where A_1, A_2 are of the form (B), then $M \equiv \lambda u.Q$, and $\Gamma(u : A_1) \vdash Q : \alpha_1$ and $\Gamma(u : A_2) \vdash Q : \alpha_2$.*

PROOF. Part (1) follows from Lemma 15.3.5 and Lemma 15.3.7. For part (2) note first that by Lemma 15.3.7(1), the term M cannot be an application. If it is a variable x, then $[\Gamma(x) \leq (A_1 \to \alpha_1) \cap (A_2 \to \alpha_2)] \in \mathcal{T}_\diamond^\eta$. This contradicts Lemma 15.3.8, because $\Gamma(x)$ must be a game type. It follows that M must be an abstraction, and the hypothesis follows from Lemma **??** 1.1.10 (vi). ∎

Tree games

In order to show the undecidability of inhabitation for $\mathcal{T}_\diamond^\eta$ ([8]) we will introduce a certain *tree game*. Tree games are an intermediate step in our construction. The idea of a tree game is to represent, in an abstract way, the crucial combinatorial behaviour of proof search in $\mathcal{T}_\diamond^\eta$. We will first show how tree games can be represented by inhabitation problems and then how tree games can represent computations of queue automata.

15.3.10. DEFINITION. Let Σ be a finite alphabet.

1. A *local move* (over Σ) is a finite nonempty set X of pairs of elements of Σ.
2. A *global move* (over Σ) is a finite nonempty set G of triples of the form
$$\langle \langle X, b \rangle, \langle Y, c \rangle, d \rangle,$$
   where $b, c, d \in \Sigma$ and X, Y are local moves.
3. A *tree game* (over Σ) is a triple of the form
$$T = \langle a, F, \{G_1, \ldots, G_n\} \rangle,$$
   where $a \in \Sigma$ is the *initial label*, $F \subseteq \Sigma$ is the set of *final labels* and $G_1, \ldots, G_n$ are global moves.

Before we explain the rules of the game, we give an interpretation of tree games in terms of types.

15.3.11. DEFINITION. Let T be a tree game over Σ, and assume that Σ is a set of type variables. Moves of T, and the set of final labels, can be interpreted as types of $\mathcal{T}_\diamond^\eta$ as follows.

[8] *Notation for $\mathcal{T}_\diamond^\eta$?*

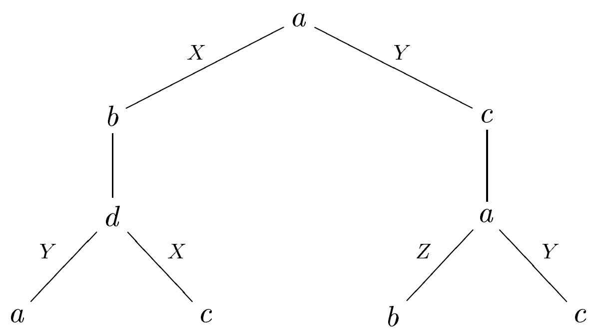

Figure 15.1: An example position

1. If $X = \{\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle\}$ then $\tilde{X} = (a_1 \to b_1) \cap \ldots \cap (a_n \to b_n)$;
2. If $G = \{\langle \langle X_1, b_1 \rangle, \langle Y_1, c_1 \rangle, d_1 \rangle, \ldots, \langle \langle X_n, b_n \rangle, \langle Y_n, c_n \rangle, d_n \rangle\}$ then $\tilde{G} = ((\tilde{X}_1 \to b_1) \cap (\tilde{Y}_1 \to c_1) \to d_1) \cap \ldots \cap ((\tilde{X}_n \to b_n) \cap (\tilde{Y}_n \to c_n) \to d_n)$.
3. If $F = \{a_1, \ldots, a_n\}$ then $\tilde{F} = a_1 \cap \ldots \cap a_n$.

A tree game is solitary game, i.e., there is only one player. Starting from an initial position, the player can nondeterministically choose a sequence of moves, and (s)he wins if (s)he can manage to reach a final position. Every position (configuration) of the game is a finite labelled tree, and at every step the depth of the tree is increasing.

15.3.12. Definition.
Let T be a tree game. A *position* of T is a finite labelled tree, satisfying the following conditions:

- The root is labelled by the initial symbol a;
- Every node has at most two children;
- Nodes at the same level (the same distance from the root) have the same number of children (in particular all leaves are at the last level); **changed**
- All nodes are labelled by elements of Σ;
- In addition, if a node v has two children v' and v'' then the branches $\langle v, v' \rangle$ and $\langle v, v'' \rangle$ are labelled by local moves.

15.3.13. Definition. Let T be a tree game.

1. The *initial position* of T consists of only one node labelled a.
2. A position $\mathcal{C}$ is *winning* iff all labels of the leaves of $\mathcal{C}$ are in F.

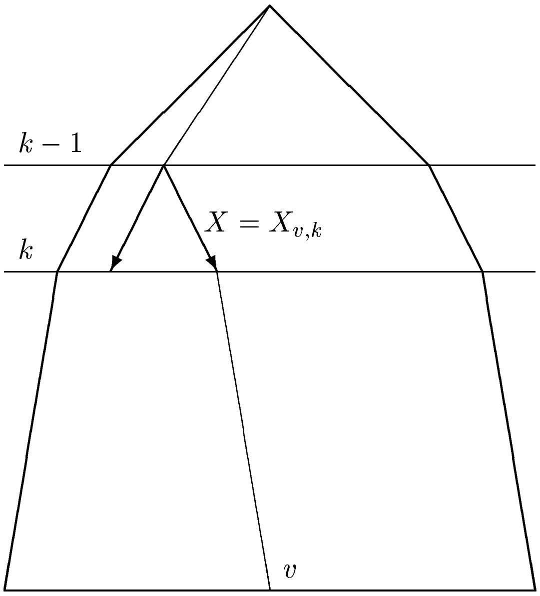

Figure 15.2: Local move X associated to node v

15.3.14. Definition. Let $\mathcal{C}$ be a position in a game $T = \langle a, F, \{G_1, \ldots, G_n\} \rangle$, and let v be a leaf of $\mathcal{C}$. Let k be such that all nodes in $\mathcal{C}$ at level $k-1$ have two children. (See Figure 15.2.) There is a node u at level $k-1$ which is an ancestor of v, one of the children of u, say u', is also an ancestor of v (possibly improper, i.e., it may happen that $u' = v$). Assume that X is the label of the branch $\langle u, u' \rangle$. Then we say that X is the *k-th local move associated to v*, and we write $X = X_{v,k}$.

Now we can finally describe the rules of the game.

15.3.15. Definition.

(i) Let $T = \langle a, F, \{G_1, \ldots, G_n\} \rangle$ and let $\mathcal{C}$ be a (current) position of T. There are two possibilities to obtain a next position.

(1) The player can perform a "global" step, by first selecting one of the global moves G_i and then performing the following actions for each leaf v of $\mathcal{C}$:
   - Choose a triple $\langle \langle X, b \rangle, \langle Y, c \rangle, d \rangle \in G_i$ such that d is the label of v;
   - Create two children of v, say v' and v'', labelled b and c, respectively;
   - Label the branch $\langle v, v' \rangle$ by X and the branch $\langle v, v'' \rangle$ by Y.

(2) The player can also perform a "local" step. This begins with a choice of a level k of $\mathcal{C}$ (each node at level $k-1$ with two children). Then, for each leaf v of $\mathcal{C}$, the player executes these actions:
   - Choose a pair $\langle a, b \rangle \in X_{v,k}$ such that b is the label of v;
   - Create a single child of v, labelled a.

(ii) If a position $\mathcal{C}'$ is reachable from $\mathcal{C}$ with help of G_i then we write $\mathcal{C} \Rightarrow^{G_i} \mathcal{C}'$. If $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a local step defined by level k, we write $\mathcal{C} \Rightarrow^{k} \mathcal{C}'$. If $\mathcal{C}'$ is reachable from $\mathcal{C}$ in one step (global or local) then we write $\mathcal{C} \Rightarrow_T \mathcal{C}'$.

(iii) Let $\mathcal{C}_0$ be the initial position. We say that the player *can win* the game iff there is a winning position $\mathcal{C}$ with $\mathcal{C}_0 \Rightarrow_T^* \mathcal{C}$.[9] [10]

15.3.16. EXAMPLE. Consider the tree game $T_0 = \langle 1, \{c\}, \{G_1, G_2\} \rangle$, over the alphabet $\Sigma = \{a, b, c, 1, 2\}$, where:

- $G_1 = \{\langle \langle \{(a,a)\}, 1 \rangle, \langle \{(b,a)\}, 2 \rangle, 1 \rangle, \langle \langle \{(a,b)\}, 1 \rangle, \langle \{(b,b)\}, 2 \rangle, 2 \rangle\}$;
- $G_2 = \{\langle \langle \{(c,a)\}, a \rangle, \langle \{(c,a)\}, a \rangle, 1 \rangle, \langle \langle \{(c,b)\}, a \rangle, \langle \{(c,b)\}, a \rangle, 2 \rangle\}$.

Figure 15.3 demonstrates a possible winning position $\mathcal{C}$ of the game. Note that this position can actually be reached from the initial one in 6 steps, so that the player can win T_0. These 6 steps are as follows

$$\mathcal{C}_0 \Rightarrow^{G_1} \mathcal{C}_1 \Rightarrow^{G_1} \mathcal{C}_2 \Rightarrow^{G_2} \mathcal{C}_3 \Rightarrow^{1} \mathcal{C}_4 \Rightarrow^{2} \mathcal{C}_5 \Rightarrow^{3} \mathcal{C}_6 = \mathcal{C},$$

where each $\mathcal{C}_i$ is of depth i. The reader should observe that every sequence of steps leading from $\mathcal{C}_0$ to a winning position must obey a similar pattern:

$$\mathcal{C}_0 \Rightarrow^{G_1} \mathcal{C}_1 \Rightarrow^{G_1} \cdots \Rightarrow^{G_1} \mathcal{C}_{n-1} \Rightarrow^{G_2} \mathcal{C}_n \Rightarrow^{1} \mathcal{C}_{n+1} \Rightarrow^{2} \cdots \Rightarrow^{n} \mathcal{C}_{2n},$$

where $\mathcal{C}_{2n}$ is winning. Thus, the game must consist of two phases: first a number of applications of G_1, then a single application of G_2 and then a sequence of steps using only local moves. What is important in our example is that the order of local steps is fully determined. Indeed, at the position $\mathcal{C}_{n+k-1}$ the only action possible is "$\Rightarrow^k$". That is, one must apply the k-th local moves associated to the leaves (the moves labelling branches at depth k). This is forced by the distribution of symbols a, b at depth $n+k-1$.

Let us emphasize a few properties of our games. First, a game is a nondeterministic process, and there are various sequences of steps possible. We can have winning sequences (reaching a winning position), and infinitely long sequences, but also "deadlocks" when no rule is applicable. Note that there are various levels of nondeterminism here: we can choose between G_i's and k's and then between various elements of the chosen set G_i (resp. $X_{v,k}$). It is an important property of the game that the actions performed at various leaves during a local step may be different, as different moves $X_{k,v}$ were "declared" before at the corresponding branches of the tree.

We now explain the relationship between tree games and term search in $\mathcal{T}_\diamond^\eta$. Since we deal with intersection types, it is not unexpected that we need sometimes to require one term to have many types, possibly within different contexts. This leads to the following definition:

[9] *Macro for double arrow?*
[10] *Exercises?*

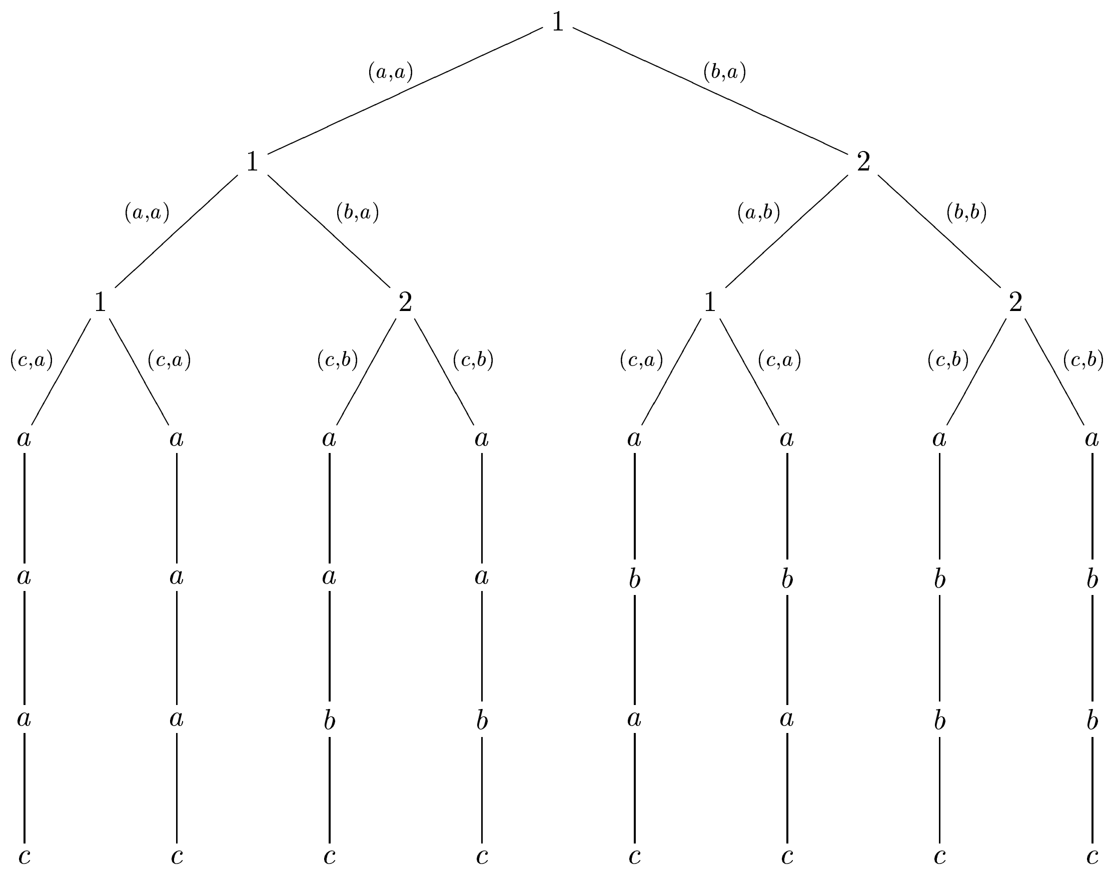

Figure 15.3: A winning position of T_0

15.3.17. DEFINITION. An instance of *generalized inhabitation problem (gip)* is a finite set of pairs $P = \{\langle \Gamma_i, \alpha_i \rangle \mid i = 1, \ldots, n\}$, where each Γ_i is a context,[11] and each α_i is a variable. (We do not need to consider arbitrary types here.) A *solution* of P is a term M such that $\Gamma_i \vdash^{\eta}_{\diamond} M : \alpha_i$ holds for each i.

15.3.18. DEFINITION. Let $T = \langle a, F, \{G_1, \ldots, G_n\} \rangle$ be a tree game. We define a basis $\Gamma_T = \{(x_0 : \tilde{F})\} \cup \{(x_i : \tilde{G}_i) \mid i = 1, \ldots, n\}$.

Now let $\mathcal{C}$ be a position of T. Let J be the set of all numbers k such that every node in $\mathcal{C}$ at level $k-1$ has two children. We associate a new variable y_k to each $k \in J$. Then we define a basis $\Gamma_v = \{(y_k : \tilde{X}_{v,k}) \mid k \in J\}$. (Recall that $X_{v,k}$ is the k-th local move associated to v, see Definition 15.3.14.)

Finally, we define an instance $P_{\mathcal{C}}$ of *gip* consisting of all pairs of the form $\langle \Gamma_T \cup \Gamma_v, a_v \rangle$, where v is a leaf of $\mathcal{C}$ labelled a_v.

The following lemma states the exact correspondence between inhabitation and games. Let us make first one comment that perhaps may help to avoid confusion. We deal here with quite a restricted form of inhabitation problem (only game[12] contexts) which implies that the lambda terms to be constructed are of a "linear" shape (Exercise 15.4.20). Thus we have to deal with trees not because of the shape of lambda terms (as often happens with proof search algorithms) but exclusively because the nature of intersection types, and because we need to solve various inhabitation problem uniformly (i.e., to solve instances of *gip*).

15.3.19. LEMMA. *Let $T = \langle a, F, \{G_1, \ldots, G_n\} \rangle$ be a tree game.*

(i) *If $\mathcal{C}$ is a winning position of T then $P_{\mathcal{C}}$ has a solution.*

(ii) *If $\mathcal{C}_1 \Rightarrow_T \mathcal{C}_2$ and $P_{\mathcal{C}_2}$ has a solution then also $P_{\mathcal{C}_1}$ has a solution.*

(iii) *If $\mathcal{C}_1$ is a non-winning position and $P_{\mathcal{C}_1}$ has a solution, then $\mathcal{C}_1 \Rightarrow_T \mathcal{C}_2$, for some $\mathcal{C}_2$ such that $P_{\mathcal{C}_2}$ has a solution.*

PROOF. (i) Obvious. (ii) Let N be a solution of $P_{\mathcal{C}_2}$. If $\mathcal{C}_1 \Rightarrow^k \mathcal{C}_2$ then $y_k N$ is a solution of $P_{\mathcal{C}_1}$. Otherwise, we have $\mathcal{C}_1 \Rightarrow^{G_i} \mathcal{C}_2$. Let k be the depth of $\mathcal{C}_2$. Then the term $x_i(\lambda y_k.N)$ is a solution of $P_{\mathcal{C}_1}$.

(iii) Observe that all contexts occurring in $P_{\mathcal{C}_i}$ are game contexts. Then use our special Generation Lemma 15.3.9 to find out that a solution M of $P_{\mathcal{C}_1}$ must either be of the form $y_k N$ or of the form $x_i(\lambda y_k.N)$ (exactly as in part (2)). Details are left to the reader. ∎

15.3.20. PROPOSITION. *Let $\mathcal{C}_0$ be the initial position of a tree game T. The player can win the game if and only if $P_{\mathcal{C}_0}$ has a solution.*

PROOF. We begin with the following claim.

Claim: *From an arbitrary position $\mathcal{C}$, the player can successfully reach a winning position if and only if the instance $P_{\mathcal{C}}$ has a solution.*

[11] *We may require that the domains of all Γ_i be equal*
[12] *game contexts?*

The claim is proved with help of Lemma 15.3.19. For the left-to-right part, one proceeds by induction on the number of steps needed to reach a winning position. For the converse, one applies induction with respect to the size of the solution.

The Proposition now follows by applying the claim to $\mathcal{C}_0$. ■

15.3.21. COROLLARY. *The problem "Can the player win a given tree game T?" is many-one reducible to the relativized inhabitation problem for $\mathcal{T}_\diamond^\eta$.*

PROOF. Immediate from Proposition 15.3.20, since $P_{\mathcal{C}_0}$ consists only of one pair. ■

Typewriters

In order to simplify our construction we introduce an auxiliary notion of a typewriter automaton. Informally, a typewriter automaton is just a reusable finite-state transducer. At each step, it reads a symbol, replaces it by a new one and changes the internal state. But at the end of the word, our automaton moves its reading and printing head back to the beginning of the tape and continues to work until a final state is reached. That is, a typewriter automaton

changed is a special case of a linear bounded automaton, cf. **?**.[13] A formal definition follows.

15.3.22. DEFINITION.

(i) A (deterministic) *typewriter automaton* $\mathcal{A}$ is a tuple of the form

$$\mathcal{A} = \langle\, \Sigma, Q, q_0, F, \varrho \,\rangle,$$

where Σ is a finite alphabet, Q is a finite set of states, $q_0 \in Q$ is an initial state

changed and $F \subseteq Q$ is a set of final states. The last component is a transition function $\varrho : (\Sigma \cup \{\varepsilon\}) \times (Q - F) \to (\Sigma \cup \{\varepsilon\}) \times Q$, which must satisfy the following condition: Whenever $\varrho(q, a) = (p, b)$ then either $a, b \in \Sigma$ or $a = b = \varepsilon$.

changed (ii) A configuration (instantaneous description) of $\mathcal{A}$ is represented by a triple $\langle\, w, q, v \,\rangle$, where (as usual) $wv \in \Sigma^*$ is the tape contents, $q \in Q$ is the current state, and one should imagine the machine head pointing at the first symbol of v.

changed (iii) The next ID function $\overline{\varrho}$ is defined as follows:

- $\overline{\varrho}(\langle\, w, q, av \,\rangle) = \langle\, wb, p, v \,\rangle$, if $a \neq \varepsilon$ and $\varrho(q, a) = (p, b)$;
- $\overline{\varrho}(\langle\, w, q, \varepsilon \,\rangle) = \langle\, \varepsilon, p, w \,\rangle$, if $\varrho(q, \varepsilon) = (p, \varepsilon)$. [14]

(iv) The language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is the set of all $w \in \Sigma^*$, such that $\overline{\varrho}^k(\langle\, \varepsilon, q_0, w \,\rangle) = \langle\, u, q, v \,\rangle$, for some k and $q \in F$.

Recall that the emptiness problem for linear bounded automata is undecidable (see **?**). We need a little stronger fact.

---

[13] *There is a new edition by Hopcroft, Motwani and Ullman. I do not remember if they treat linear bounded automata.*

[14] Standard notation: ε, Σ^*.

15.3.23. LEMMA. *The emptiness problem for typewriter automata ("Given $\mathcal{A}$, is $L(\mathcal{A})$ empty or not?") is undecidable.*

PROOF. Exercise 15.4.22 ■

Our goal is now to represent typewriters as games. We begin with a refinement of Example 15.3.16. In the following definitions, triples of the form $\langle \langle X, b \rangle, \langle Y, c \rangle, d \rangle$ will be represented graphically as follows:

$$\begin{tikzcd} & d \arrow[dl, "X"'] \arrow[dr, "Y"] & \\ b & & c \end{tikzcd}$$

This is necessary to enhance readability.

15.3.24. DEFINITION. The alphabet Σ_1 is a Cartesian product of the form:

$$\Sigma_1 = \{\bot, a, b\} \times \{\bot, 1, 2\} \times \{L, G\}.$$

We define a tree game $T_1 = \langle \langle \bot, 1, G \rangle, \emptyset, \{G_1, G_2, G_3\} \rangle$. (The set of accepting labels is $\emptyset$, because we are interested in all possible plays of the game) The moves are defined as follows.

(i) Global move G_1 consists of the following two triples:

$$\begin{tikzcd} & \langle \bot, 1, G \rangle \arrow[dl, "{\{(\langle a,x,G \rangle,\langle a,x,L \rangle) \mid \text{all } x\}}"'] \arrow[dr, "{\{(\langle b,x,G \rangle,\langle a,x,L \rangle) \mid \text{all } x\}}"] & \\ \langle \bot, 1, G \rangle & & \langle \bot, 2, G \rangle \end{tikzcd}$$

and:

$$\begin{tikzcd} & \langle \bot, 2, G \rangle \arrow[dl, "{\{(\langle a,x,G \rangle,\langle b,x,L \rangle) \mid \text{all } x\}}"'] \arrow[dr, "{\{(\langle b,x,G \rangle,\langle b,x,L \rangle) \mid \text{all } x\}}"] & \\ \langle \bot, 1, G \rangle & & \langle \bot, 2, G \rangle \end{tikzcd}$$

Before we define other global moves, let us point out that G_1 is very similar to rule G_1 of the game T_0 in Example 15.3.16 (observe the a and b in the first **changed**
component and 1 and 2 in the second one).

(ii) Global move G_2 consists again of two triples:

$$\begin{tikzcd}
 & \langle \perp, 1, G \rangle \arrow[dl, swap, "\{(\langle a,x,G,\rangle,\langle a,x,L\rangle) \mid \text{all } x\}"] \arrow[dr, "\{(\langle b,x,G,\rangle,\langle a,x,L\rangle) \mid \text{all } x\}"] & \\
\langle a, 1, L \rangle & & \langle a, 2, L \rangle
\end{tikzcd}$$

and:

$$\begin{tikzcd}
 & \langle \perp, 2, G \rangle \arrow[dl, swap, "\{(\langle a,x,G,\rangle,\langle b,x,L\rangle) \mid \text{all } x\}"] \arrow[dr, "\{(\langle b,x,G,\rangle,\langle b,x,L\rangle) \mid \text{all } x\}"] & \\
\langle a, 1, L \rangle & & \langle a, 2, L \rangle
\end{tikzcd}$$

(iii) Global move G_3 consists of triples of the form (where $z \in \{a, b\}$, i.e., $z \neq \perp$):

$$\begin{tikzcd}
 & \langle z, 1, G \rangle \arrow[dl, swap, "\{(\langle a,x,G\rangle,\langle a,x,L\rangle) \mid \text{all } x\}"] \arrow[dr, "\{(\langle b,x,G\rangle,\langle a,x,L\rangle) \mid \text{all } x\}"] & \\
\langle z, 1, L \rangle & & \langle z, 2, L \rangle
\end{tikzcd}$$

and of the form

$$\begin{tikzcd}
 & \langle z, 2, G \rangle \arrow[dl, swap, "\{(\langle a,x,G\rangle,\langle b,x,L\rangle) \mid \text{all } x\}"] \arrow[dr, "\{(\langle b,x,G\rangle,\langle b,x,L\rangle) \mid \text{all } x\}"] & \\
\langle z, 1, L \rangle & & \langle z, 2, L \rangle
\end{tikzcd}$$

15.3.25. Lemma. *Every sequence of positions reachable from the initial position $\mathcal{C}_0$ of T_1 must be of the form:*

$$\begin{gathered}\mathcal{C}_0 \Rightarrow^{G_1} \mathcal{C}_1 \Rightarrow^{G_1} \cdots \Rightarrow^{G_1} \mathcal{C}_i \Rightarrow^{G_1} \mathcal{C}_{i+1} \Rightarrow^{G_1} \mathcal{C}_{i+2} \Rightarrow^{G_1} \cdots \Rightarrow^{G_1} \mathcal{C}_{n-1} \Rightarrow^{G_2} \\ \mathcal{C}_n \Rightarrow^{1} \mathcal{C}_{n+1} \Rightarrow^{G_3} \Rightarrow^{2} \cdots \Rightarrow^{G_3} \mathcal{C}_{n+2i} \Rightarrow^{i+1} \mathcal{C}_{n+2i+1} \Rightarrow^{G_3} \mathcal{C}_{n+2i+2} \Rightarrow^{i+2} \cdots \\ \cdots \Rightarrow^{G_3} \mathcal{C}_{3n-2} \Rightarrow^{n} \mathcal{C}_{3n-1} \Rightarrow^{G_3} \mathcal{C}_{3n} \Rightarrow^{n+2} \mathcal{C}_{3n+1} \Rightarrow^{G_3} \mathcal{C}_{3n+2} \Rightarrow^{n+4} \mathcal{C}_{3n+3} \Rightarrow^{G_3} \cdots \\ \cdots \Rightarrow^{G_3} \mathcal{C}_{2i} \Rightarrow^{2i-n+2} \mathcal{C}_{2i+1} \Rightarrow^{G_3} \cdots\end{gathered}$$

That is, starting from the n-th position, the local and global steps alternate, and the local moves declared at every second step are executed exactly after $n-2$ steps.

Proof. Exercise 15.4.23. ∎

15.3.26. Definition. Let T_1 be as above and let $\mathcal{A} = \langle\, \Sigma^{\mathcal{A}}, Q, q_0, F^{\mathcal{A}}, \varrho \,\rangle$ be a typewriter automaton. We define a game $T^{\mathcal{A}}$ as follows[15]

(i) The alphabet of $T^{\mathcal{A}}$ is the Cartesian product $\Sigma_1 \times (\Sigma^{\mathcal{A}} \cup \{\bot, \varepsilon\}) \times Q$. **changed**

(ii) For each local move X of T and each $\beta \in \Sigma^{\mathcal{A}} \cup \{\varepsilon\}$, we define a local move **changed** $X_\beta = \{(\langle\, a, \beta, q \,\rangle, \langle\, b, \bot, q \,\rangle \mid q \in Q \text{and } \langle\, a, b \,\rangle \in X\}$.

(iii) For each triple $\Delta = \langle\, \langle\, X, b \,\rangle, \langle\, Y, c \,\rangle, d \,\rangle \in G_3$ we define a set $\Delta^{\mathcal{A}}$ consisting of all triples of the form **changed**

$$\begin{tikzcd}[column sep=small] & (d,\alpha,q) \arrow[dl, "X_\beta"'] \arrow[dr, "Y_\beta"] & \\ (b,\bot,p) & & (c,\bot,p) \end{tikzcd}$$

where $\varrho(\alpha, q) = (\beta, p)$.

Alternative: ...we define a set[16] **changed**
$\Delta^{\mathcal{A}} = \{\langle\, \langle\, X_\beta, (b, \bot, p) \,\rangle, \langle\, Y_\beta, (c, \bot, p) \,\rangle, (d, \alpha, q) \,\rangle \mid \alpha, \beta \in \Sigma^{\mathcal{A}} \cup \{\varepsilon\}, p, q \in Q \text{and } \varrho(\alpha, q) = (\beta, p)\}$
changed

(iv) If $\Delta = \langle\, \langle\, X, b \,\rangle, \langle\, Y, c \,\rangle, d \,\rangle \in G_1 \cup G_2$ then we define Δ_β as the triple

$$\begin{tikzcd}[column sep=small] & (d,\bot,q_0) \arrow[dl, "X_\beta"'] \arrow[dr, "Y_\beta"] & \\ (b,\bot,q_0) & & (c,\bot,q_0) \end{tikzcd}$$

changed

(v) For each $\beta \in \Sigma_{\mathcal{A}}$, we take $G_1^{\mathcal{A}}(\beta) = \{\Delta_\beta \mid \Delta \in G_1\}$;

(vi) Then we define $G_2^{\mathcal{A}} = \{\Delta_\varepsilon \mid \Delta \in G_2\}$ and $G_3^{\mathcal{A}} = \bigcup\{\Delta^{\mathcal{A}} \mid \Delta \in G_3\}$.

(vii) The initial symbol of $T^{\mathcal{A}}$ is $a = \langle\, a_0, \bot, q_0 \,\rangle$, where a_0 is the initial symbol of T_1. **changed**

(viii) The set of final symbols is $F = \Sigma_1 \times (\Sigma_A \cup \{\bot, \varepsilon\}) \times F^{\mathcal{A}}$;

(ix) Finally, we take $T^{\mathcal{A}} = \langle\, a, F, \{G_1^{\mathcal{A}}(\beta) \mid \beta \in \Sigma_{\mathcal{A}}\} \cup \{G_2^{\mathcal{A}}, G_3^{\mathcal{A}}\} \,\rangle$

15.3.27. Proposition. *For every typewriter automaton $\mathcal{A}$, the player can win $T^{\mathcal{A}}$ if and only if the language $L(\mathcal{A})$ is nonempty.*

changed

PROOF. Our game $T^{\mathcal{A}}$ behaves as a "Cartesian product" of T_1 and $\mathcal{A}$. Informally speaking, there is no communication between the first component and the other two. In particular:

changed

1. Lemma 15.3.25 remains true with T_1 replaced by $T^{\mathcal{A}}$ and G_1, G_2, G_3 replaced respectively by $G_1^{\mathcal{A}}(\beta)$, $G_2^{\mathcal{A}}$ and $G_3^{\mathcal{A}}$. That is a legitimate sequence of positions must look as follows:

changed

$$\mathcal{C}_0 \Rightarrow^{G_1^{\mathcal{A}}(\beta_1)} \mathcal{C}_1 \Rightarrow^{G_1^{\mathcal{A}}(\beta_2)} \dots \Rightarrow^{G_1^{\mathcal{A}}(\beta_{n-1})} \mathcal{C}_{n-1} \Rightarrow^{G_2^{\mathcal{A}}}$$
$$\mathcal{C}_n \Rightarrow^{1} \mathcal{C}_{n+1} \Rightarrow^{G_3^{\mathcal{A}}} \Rightarrow^{2} \cdots \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{n+2i} \Rightarrow^{i+1} \mathcal{C}_{n+2i+1} \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{n+2i+2} \Rightarrow^{i+2} \cdots$$
$$\cdots \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{3n-2} \Rightarrow^{n} \mathcal{C}_{3n-1} \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{3n} \Rightarrow^{n+2} \mathcal{C}_{3n+1} \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{3n+2} \Rightarrow^{n+4} \mathcal{C}_{3n+3} \Rightarrow^{G_3^{\mathcal{A}}} \cdots$$
$$\cdots \Rightarrow^{G_3^{\mathcal{A}}} \mathcal{C}_{2i} \Rightarrow^{2i-n+2} \mathcal{C}_{2i+1} \Rightarrow^{G_3^{\mathcal{A}}} \cdots$$

2. If a position $\mathcal{C}$ of $\mathcal{T}^{\mathcal{A}}$ can be reached from the initial position, then the second and third component of labels are always the same for all nodes at every fixed level of $\mathcal{C}$.

changed

Consider a sequence of positions $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots$ as in (i) above. Note that this sequence is fully determined by the choice of n and $\beta^1, \dots, \beta^{n-1}$. (Also observe that β^i, for $i = 1, \dots, n-1$ are the second components of the labels of all leaves of $\mathcal{C}_{n+2i-1}$.) Let w denote the word $\beta^1\beta^2\dots\beta^{n-1}$.

changed

Claim: The automaton $\mathcal{A}$ accepts w if and only if $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots$ leads to a winning position.

changed

We shall now prove the claim. Let β_k^j denotes the symbol contained in the j-th cell of the tape of $\mathcal{A}$, after the machine has completed $k-1$ full phases (the tape has been fully scanned $k-1$ times). That is, β_k^j is the symbol to be read during the k-th phase. Of course β_1^j is β^j. For uniformity, we also define $\beta_k^n = \varepsilon$. Further, let $q_{k,j}$ be the internal state of the machine, just before it reads the j-th cell for the k-th time (i.e., after $k-1$ full phases). The reader will easily show that for all k and all $j = 1, \dots, n$

changed (i) The second component of labels of all leaves of $\mathcal{C}_{(2k-1)n+2j-1}$ is β_k^j,

changed (ii) The third component of labels of all leaves of $\mathcal{C}_{(2k-1)n+2j-2}$ is $q_{k,j}$.

new

The claim follows when (ii) is applied to the final states. Figure 15.4 illustrates the induction hypothesis by showing labels of a node at depth $(2k-1)n+2j-2$ together with her daughter and grandchildren. ∎

15.3.28. THEOREM. *The inhabitation problem for $\mathcal{T}_\diamond^\eta$ and $\mathcal{T}_\diamond^{\Omega\eta}$ is undecidable.*

changed

PROOF. For $\mathcal{T}_\diamond^\eta$, the result follows from Corollary 15.3.21, Lemma 15.3.23, Proposition 15.3.27. For $\mathcal{T}_\diamond^{\Omega\eta}$ one uses Lemma 15.3.2. ∎

[15] *does not this* `\sub` *look ugly?*

[16] *choose one*

$$\begin{tikzcd}
 & (a, \perp, q_{k,j}) \arrow[d] & \\
 & (b, \beta^j_k, q_{k,j}) \arrow[dl, "X_{\beta^j_{k+1}}"'] \arrow[dr, "Y_{\beta^j_{k+1}}"] & \\
(c, \perp, q_{k,j+1}) & & (d, \perp, q_{k,j+1})
\end{tikzcd}$$

Figure 15.4: Simulation of a single machine step

Remarks

The proof of undecidability of the inhabitation problem presented in this section is a modified version of the original proof in **?**. It should be observed that all types in game contexts are of rank at most 3 in the sense of Leivant **?**. (That is, intersections may occur in arguments of arguments of types.)

Variant: (Recall that $\mathit{rank}(\tau) = 0$, for simple types τ, and otherwise $\mathit{rank}(\tau \cap \sigma) = \max(1, \mathit{rank}(\tau), \mathit{rank}(\sigma))$, and $\mathit{rank}(\tau \to \sigma) = \max(1 + \mathit{rank}(\tau), \mathit{rank}(\sigma))$, if $\cap$ occurs in $\tau \to \sigma$.)

Thus, the relativized inhabitation problem is undecidable for contexts of rank 3, **changed**
and the inhabitation problem (with empty contexts) is undecidable for types of rank four. It is an open problem whether inhabitation is decidable for rank 3. **changed**
But it is decidable for rank 2 (Exercise 15.4.24). Some other decidable cases are discussed in **?**.

From the point of view of the the *formulae as types* principle, (the "Curry-Howard isomorphism"), the inhabitation problem should correspond to a provability problem for a certain logic. It is however not at all obvious what should be the logic corresponding to intersection types. We deal here with a "proof-functional" rather than "truth-functional" connective $\cap$, called sometimes "strong conjunction": a proof of $A \cap B$ must *be* a proof of both A and B, **changed**
rather than merely *contain* two separate proofs of A and B. See **?** for the discussion of strong conjunction logics.

Various authors attempted to define Church-style calculi for intersection types, which is another way leading to understand the logic of intersection types. We refer the reader to **?** for this approach.[17]

15.4. Exercises 26.2.2004:1018

15.4.1. Using the generation lemmata show the following.

[17] *Mariangiola, will you expand this remark? Refer to the remainder of this section?*

(i) $\nvdash_B^{\mathrm{CD}} \mathbf{1} : \psi \to \psi$, where ψ is any constant.
(ii) $\nvdash_B^{\mathrm{HL}} \mathbf{K} : \omega$.
(iii) $x{:}\nu \nvdash^{\mathrm{EHR}\Omega} \mathbf{K}x(yz) : \nu$.
(iv) $\nvdash^{\mathrm{Scott}\Omega} \mathbf{I} : \omega$.
(v) $\nvdash^{\mathrm{Plotkin}\Omega} \mathbf{I}x : \omega$.

15.4.2. Show the following.
(i) $y{:}\Omega \vdash^{\mathrm{AO}} (\lambda x.xy) : \Omega \to \Omega$. [Hint. Use ($\Omega$-lazy).]
(ii) $\Omega \not\leq_{\mathrm{AO}} (\Omega \to \Omega)$. [Hint. Prove by induction

$$A \leq_{\mathrm{AO}} (P \to Q) \cap (\bigcap_i R_i) \quad \Rightarrow \quad A \equiv (B \to C) \cap (\bigcap_j D_j),$$

for some $B, C, \vec{D}$.]

(iii) $y{:}\Omega, x{:}\Omega \nvdash^{\mathrm{AO}} xy : \Omega$.
(iv) Conclude that for $\nabla = \mathrm{AO}$ one has

$$\Gamma \vdash^{\nabla} (\lambda x.M) : (B \to \Omega) \not\Rightarrow \Gamma, x{:}B \vdash^{\nabla} M : \Omega.$$

(v) Conclude that $\lambda\cap^{\mathrm{AO}} \not\models \beta$-red. [Hint. Use (iv) and Proposition 15.2.2(i).]

15.4.3. Show that $\nabla^{\mathrm{Plotkin}}$ is a beta theory by checking that it satisfies the following stronger condition:

$$\begin{aligned} &\textstyle\bigcap_{i\in I}(A_i \to B_i) \leq_{\mathrm{Plotkin}} C \to D \;\Rightarrow \\ &\exists i{\in}I,\ C \sim_{\mathrm{Plotkin}} A_i \,\&\, B_i \sim_{\mathrm{Plotkin}} D. \end{aligned}$$

15.4.4. Show that $\nabla^{\mathrm{Engeler}}$ is a beta theory by checking that it satisfies the following stronger condition:

$$\begin{aligned} &\textstyle\bigcap_{i\in I}(A_i \to B_i) \leq_{\mathrm{Engeler}} C \to D \,\&\, D \not\sim_{\nabla} \Omega \;\Rightarrow \\ &\exists J \subseteq I.\ (\forall i{\in}J.\ C \sim_{\mathrm{Engeler}} A_i) \;\&\; \textstyle\bigcap_{i\in J} B_i \sim_{\mathrm{Engeler}} D. \end{aligned}$$

15.4.5. Prove that Theorem 15.1.16(ii) holds also under the weaker conditions:

$$A \leq_{\nabla} B \;\Rightarrow\; C \to A \leq_{\nabla} C \to B$$

$$(A \to B) \cap (C \to D) \leq_{\nabla} A \cap C \to B \cap D.$$

15.4.6. (i) Let $\vdash$ denote derivability in the system obtained from $\vdash^{\mathrm{CD}}$ by replacing rule ($\leq_{\mathrm{CD}}$) by the rules

$$(\cap\mathrm{E}) \quad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B}.$$

Then

$$\Gamma \vdash^{\mathrm{CD}} M : A \;\Leftrightarrow\; \Gamma \vdash M : A.$$

(ii) Let $\vdash$ denote derivability in the system obtained from $\vdash^{\mathrm{CDV}}$ by replacing rule $(\leq_{\mathrm{CDV}})$ by the rules $(\cap\mathrm{E})$ and

$$(R\eta) \quad \frac{\Gamma \vdash \lambda x.Mx : A}{\Gamma \vdash M : A} \quad \text{if } x \notin \mathrm{FV}(M).$$

Then

$$\Gamma \vdash^{\mathrm{CDV}} M : A \;\Leftrightarrow\; \Gamma \vdash M : A.$$

15.4.7. (Barendregt et al. [1983]) Let $\vdash$ denote derivability in the system obtained from $\vdash^{\mathrm{BCD}}_{\Omega}$ by replacing rule $(\leq_{\mathrm{BCD}})$ by the rules $(\cap\mathrm{E})$ and $(R\eta)$ as defined in exercise 15.4.6. Verify that $\Gamma \vdash^{\mathrm{BCD}}_{\Omega} M : A$ iff $\Gamma \vdash M : A$.

15.4.8. Show that rules (β-*red*) and (β-*exp*) are admissible in the system $\mathcal{D}\Omega$ as defined in Exercise 14.3.9.

15.4.9. Show that rules (η-*red*) and (η-*exp*) are not admissible in the system $\mathcal{D}\Omega$ as defined in Exercise 14.3.9.

15.4.10. Verify that if $\triangledown \in \{\mathrm{CD}, \mathrm{CDV}, \mathrm{BCD}, \mathrm{Plotkin}, \mathrm{Engeler}\}$, then $\nabla^{\triangledown}$ is not an eta theory.

15.4.11. Show that the ν-soundness of $\nabla^{\triangledown}$ is a necessary condition for the admissibility of rule (β-*red*) in $\lambda\cap^{\triangledown}_{\nu}$.

15.4.12. Show that the following condition

$$B \sim_{\triangledown} \Omega \text{ whenever } A \to B \sim_{\triangledown} \Omega \to \Omega$$

is necessary for the admissibility of rule (β-*red*) in $\lambda\cap^{\triangledown}$.

15.4.13. Rule (β_v-exp$''$) is the restriction of (β_v-exp) to bases such that $\Gamma \supseteq \Gamma_{\nu} = \{x{:}\nu \mid x \in \mathsf{V}\}$ with $\nu \in \mathbb{C}^{\triangledown}$, where $\triangledown = \mathrm{EHR}$[18].

(i) Show that $\lambda\cap^{\nu} \models (\beta_v\text{-exp}'')$.

(ii) Show that $\beta\mathsf{I}$-exp, β_v-exp and $\beta\mathsf{N}$-exp are not valid in general. [Hint. Consider $\mathsf{K}\Omega$ and $\mathsf{K}_* x$.]

15.4.14. Show that if ∇ satisfies $(\to\cap)$, (η) and $(\Omega\eta)$, then

$$\mathcal{T}^{\nabla} \text{ is } \eta\Omega\text{-sound} \;\Leftrightarrow\; \forall A \in \mathbb{C}\, \exists \vec{B}, \vec{C}\; A \sim_{\triangledown} (B_1 \to C_1) \cap \ldots \cap (B_n \to C_n).$$

15.4.15. (i) Show that for $\nabla \in \{\mathrm{BCD}, \mathrm{CDV}, \mathrm{Engeler}, \mathrm{Plotkin}, \mathrm{EHR}, \mathrm{CD}\}$ one has

$$\lambda\cap^{\nabla} \not\models (\eta\text{-exp}).$$

(ii) Show that for $\nabla \in \{\mathrm{BCD}, \mathrm{Engeler}, \mathrm{Plotkin}\}$ one has

$$\lambda\cap^{\nabla\Omega} \not\models (\eta\text{-exp}).$$

15.4.16. Notice that there are β-redexes that are typable whenever their contracta are, without being $\beta\mathbf{I}$-, $\beta\mathsf{N}$- or β_v-redex. [Hint. Consider $(\lambda x.(yz))(yz)$.]

15.4.17. Prove that the problem $? \vdash^{\triangledown}_{\diamond}\ ? : A$ is decidable. Hint: this one is obvious.

[18] This restriction is sensible for the call-by-value λ-calculus where type variables are values, see Section 17.3.

15.4.18. Prove Lemma 15.3.3. Hint: proceed by induction with respect to the definition of $\leq$. But first choose an appropriate induction hypothesis.

15.4.19. Consider a type assignment system without rule $(\leq)$ and with or without the constant ω (and rule (Ax-Ω)).

- Prove an analogue of Lemma 15.3.2.
- Let $\vdash$ stands for type assignment without rules $(\leq)$ and (Ax-Ω). Prove that if Γ is a game context then $\Gamma \vdash^{\eta}_{\diamond} M : \alpha$ is equivalent to $\Gamma \vdash M : \alpha$, for all type variables α. Conclude that type inhabitation remains undecidable without $(\leq)$.
- Prove that the type $\delta \cap (\alpha \to \beta) \cap (\alpha \to \gamma) \to \delta \cap (\alpha \to \beta \cap \gamma)$ is inhabited in $\mathcal{T}^{\eta}_{\diamond}$ but is not inhabited without rule $(\leq)$.

15.4.20. Let Γ be a game context and α be a type variable. Prove that if $\Gamma \vdash^{\eta}_{\diamond} M : \alpha$ then the Böhm tree of M has no branching.

15.4.21. Complete the proofs of Proposition 15.3.20 and Lemma 15.3.21.

15.4.22. Prove Lemma 15.3.23. Hint: Encode a queue automaton (also called a Post machine, see **?**) into a typewriter, thus reducing the halting problem for queue automata to the emptiness problem for typewriters. One possible way of doing it is as follows. Represent a queue, say "011100010", as a string of the form "$\$\$\cdots\$<011100010>\sharp\cdots\sharp\sharp$" with a certain number of the \$'s and $\sharp$'s. The initial empty queue is just "$<>\sharp\cdots\sharp\sharp$". Now an *insert* instruction means: *replace "$>$" with a digit and replace the first "$\sharp$" with "$>$"*, and similarly for a *remove*. The number of \$'s increases after each *remove*, while the suffix of $\sharp$'s shrinks after each *insert*, so that the queue "moves to the right". If the number of the initial suffix of $\sharp$'s is sufficiently large, then a typewriter automaton can verify the queue computation.

15.4.23. Prove Lemma 15.3.25. **Hint:** Compare T_1 to the game T_0 of Example 15.3.16. Observe that in each sequence of positions $\mathcal{C}_{(2k+1)n}, \ldots, \mathcal{C}_{(2k+3)n}$, the odd steps behave as an initial phase of T_0, while the even steps behave as a final phase of T_1.

15.4.24. Prove that the inhabitation problem for types of rank 2 is decidable. (A type is of rank 2 iff it is either a simple type or a type of the form $A_1 \cap \cdots \cap A_n \to A_0$, where all A_i are simple types.) **Hint:** Generalize the proof for simple types.[19]

[19] *Reference.*

Chapter 16

Models $_{26.2.2004:1018}$

16.1. Meet semi-lattices and algebraic lattices

16.1.1. DEFINITION. (i) A *meet semi-lattice* is a partial order $\mathcal{S} = \langle S, \leq, \cap \rangle$ such that for $s, t \in S$ the element $s \cap t$ is the greatest lower bound of s and t.

(ii) A meet semi-lattice *with top* is a partial order $\mathcal{S} = \langle S, \leq, \cap, \top \rangle$ such that $\forall s \in S. s \leq \top$.

The following definition is due to Scott. [Explain a bit more.]

16.1.2. DEFINITION. (i) Let $\mathcal{S}, \mathcal{S}'$ be meet semi-lattices with top. A relation $\mu \subseteq S \times S'$ is an *approximable mapping* between $\mathcal{S}$ and $\mathcal{S}'$, notation $\mathcal{S} - \mu - \mathcal{S}'$, if for all $s, t \in \mathcal{S}$ and $s', t', t'_1, t'_2 \in \mathcal{S}'$ one has

$$\begin{array}{ll} (a) & \top\ \mu\ \top'; \\ (b) & t \leq s\ \mu\ s' \leq t' \ \Rightarrow\ t\ \mu\ t'; \\ (c) & s\ \mu\ t'_1\ \&\ s\ \mu\ t'_2 \ \Rightarrow\ s\ \mu\ (t'_1 \cap t'_2). \end{array}$$

(ii) The category **MSL** has as objects the (at most) *countable* meet semi-lattices with top and as morphisms approximable mappings. Composition $\mu' \circ \mu \in \mathbf{MSL}(S, S'')$ of $\mu \in \mathbf{MSL}(S, S')$, $\mu' \in \mathbf{MSL}(S', S'')$ is the relational composition: $\mu' \circ \mu = \mu \circ_r \mu'$ with

$$s\,(\mu \circ_r \mu')\,s'' \ \Leftrightarrow\ \exists s'.\, s\,\mu\,s'\mu'' s''.$$

The order switch is due to the convention for functions: $g \circ f$ first applies f, then g. The relation $\leq\ \subseteq \mathcal{S} \times \mathcal{S}$ clearly is the identity morphism $\mathbf{1}_{\mathcal{S}} \in \mathbf{MSL}(\mathcal{S}, \mathcal{S})$.

16.1.3. DEFINITION. (i) $\mathbf{MSL}_0$ is the category with the same objects as **MSL** but as morphisms maps preserving $\leq, \cap$ and $\top$.

(ii) For $f \in \mathbf{MSL}_0(\mathcal{S}, \mathcal{S}')$ we define $\mu_f \in \mathbf{MSL}(\mathcal{S}, \mathcal{S}')$ by

$$s\ \mu_f\ s' \ \Leftrightarrow\ f(s) \leq s'.$$

16.1.4. LEMMA. *Let $\mu \in \mathbf{MSL}(\mathcal{S}, \mathcal{S}')$ be an isomorphism with inverse ν. Then there exist $f, g \in \mathbf{MSL}_0(\mathcal{S}, \mathcal{S}')$ such that f is an $\mathbf{MSL}_0$ isomorphism with inverse g and $\mu = \mu_f$, $\nu = \mu_g$.*

PROOF. By assumption $\mu \circ_r \nu = \leq$, $\nu \circ_r \mu = \leq'$.

Claim. For all $s \in \mathcal{S}$ there exists a $s' \in \mathcal{S}'$ such that

$$\begin{array}{ll} s\,\mu\,s'\,\nu\,s & (1_{s'}) \\ \forall t' \in \mathcal{S}'[s\,\mu\,t' \;\Leftrightarrow\; s' \leq' t'] & (2_{s'}) \end{array}$$

Indeed, since $s \leq s$, so $s\,\mu \circ_r \nu\,s$ one can find s' satisfying (1). From this (2) follows:

$$\begin{array}{rcl} s\,\mu\,t' & \Rightarrow & s'\,\nu\,s\,\mu\,t' \\ & \Rightarrow & s' \leq' t' \\ & \Rightarrow & s\,\mu\,s' \leq' t' \\ & \Rightarrow & s\,\mu\,t' \end{array}$$

We moreover claim that the s' above is unique. Suppose both s'_1, s'_2 satisfy (1), (2) above. Then

$$\left.\begin{array}{l} s\,\mu\,s'_1 \\ s\,\mu\,s'_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} s'_1 \leq' s'_2 & \text{by } (2_{s'_1}) \\ s'_2 \leq' s'_1 & \text{by } (2_{s'_2}) \end{array}\right.$$

hence $s'_1 = s'_2$. So we can define $f : \mathcal{S} \to \mathcal{S}'$ by $f(s) = s'$ satisfying (1), (2) above. Then f is monotonic. Indeed, if $s_0 \leq s$, then as $s\,\mu\,f(s)$, one has $s_0\,\mu\,f(s)$, since μ is an approximable mapping. But then $f(s_0) \leq' f(s)$ by $(2_{f(s_0)})$. Note that by $(2_{f(s)})$ we have $\mu = \mu_f$.

Similarly one can define a monotonic $g : \mathcal{S}' \to \mathcal{S}$ such that $\nu = \mu_g$. These f, g will constitute an $\mathbf{MSL}_0$ isomorphism. We have (using a renaming of variables) for $s \in \mathcal{S}, s' \in \mathcal{S}'$

$$\begin{array}{ll} s\,\mu\,f(s)\,\nu\,s & (a) \\ s'\,\nu\,g(s')\,\mu\,s' & (b) \\ s\,\mu\,s' \;\Leftrightarrow\; f(s) \leq' s' & (c) \\ s'\,\nu\,s \;\Leftrightarrow\; g(s') \leq s & (d) \end{array}$$

Then $g(f(s)) \leq s$, by (a), (d), and $s'\,\nu\,g(s')$ by (b). Taking $s' = f(s)$ and using (a) we get

$$\begin{array}{l} s\,\mu\,f(s)\,\nu\,g(f(s)) \\ \Rightarrow \quad s\,\mu \circ_r \nu\,g(f(s)) \\ \Rightarrow \quad s \leq g(f(s)). \end{array}$$

So we have $s = g(f(s))$, for all $s \in \mathcal{S}$. Similarly one has $s' = f(g(s'))$, for $s' \in \mathcal{S}'$.

We have seen that f, g are bijections and since they preserve the partial order, they also preserve the top and intersection. ■

16.1.5. COROLLARY. *Let $\mathcal{S}, \mathcal{S}'$ be two objects in* **MSL**. *Then*

$$\mathcal{S} \cong \mathcal{S}' \text{ in } \mathbf{MSL}_0 \;\Leftrightarrow\; \mathcal{S} \cong \mathcal{S}' \text{ in } \mathbf{MSL}.$$

PROOF. ($\Rightarrow$) One has $\mu_g \circ \mu_f = \mu_{g \circ f}$. Now if f, g constitute an isomorphism in $\mathbf{MSL}_0$, then both $f \circ g$ and $g \circ f$ are the identity, hence

$$\mu_f \circ \mu_g = \mu_{f \circ g} = \mu_I = \leq,$$

and similarly $\mu_g \circ \mu_f = \leq$, are the identity in **MSL**.

($\Leftarrow$). By the preceding lemma. ■

16.1.6. DEFINITION. (i) A *complete lattice* is a poset $\mathcal{D} = (D, \sqsubseteq)$ such that for arbitrary $X \subseteq D$ the supremum $\bigsqcup X \in D$ exists. Then one has also a *top* element $\top = \bigsqcup D$, a *bottom* element $\bot = \bigsqcup \emptyset$, arbitrary *infima*

$$\bigsqcap X = \bigsqcup \{z \mid \forall x \in X.z \sqsubseteq x\}$$

and the *sup* and *inf* of two elements

$$x \sqcup y = \bigsqcup \{x, y\}, \quad x \sqcap y = \bigsqcap \{x, y\}.$$

(ii) A subset $Z \subseteq D$ is called *directed* if Z is non-empty and

$$\forall x, y \in Z \exists z \in Z.x \sqsubseteq z \ \& \ y \sqsubseteq z.$$

(iii) An element $d \in D$ is called *compact* (or *finite*) if for every directed $Z \subseteq D$ one has

$$d \sqsubseteq \bigsqcup Z \ \Rightarrow \ \exists z \in Z.d \sqsubseteq z.$$

Note that if d, e are compact, then so is $d \sqcup e$[1]. The set of compact elements of D is denoted by $\mathcal{K}(\mathcal{D})$.

(iv) $\mathcal{D}$ is called an *algebraic lattice* if

$$\forall x \in D.x = \bigsqcup \{e \in \mathcal{K}(\mathcal{D}) \mid e \sqsubseteq x\}.$$

$\mathcal{D}$ is called an ω-*algebraic lattice* if moreover $\mathcal{K}(\mathcal{D})$ is countable (finite or countably infinite).

Instead of $d \in D$ ($X \subseteq D$) we often write $d \in \mathcal{D}$ ($X \subseteq \mathcal{D}$ respectively).

The following links the notion of a compact element to the notion of compact subset of a topological space.

16.1.7. LEMMA. *Let $\mathcal{D}$ be a complete lattice. Then $d \in \mathcal{D}$ is compact iff*

$$\forall Z \subseteq \mathcal{D}.[d \sqsubseteq Z \ \Rightarrow \ \exists Z_0 \subseteq Z.[Z_0 \textit{ is finite } \ \& \ d \sqsubseteq \bigsqcup Z_0]].$$

PROOF. ($\Rightarrow$) Suppose $d \in \mathcal{D}$ is compact. Given $Z \subseteq \mathcal{D}$, let

$$Z^+ = \{\bigsqcup Z_0 \mid Z_0 \subseteq Z \ \& \ Z_0 \text{ finite}\}.$$

Then $Z \subseteq Z^+, \bigsqcup Z_0 = \bigsqcup Z$ and Z^+ is directed. Hence

$$\begin{aligned} d \sqsubseteq \bigsqcup Z \quad & \Rightarrow \quad d \sqsubseteq \bigsqcup Z^+ \\ & \Rightarrow \quad \exists z^+ \in Z^+.d \sqsubseteq z^+ \\ & \Rightarrow \quad \exists Z_0 \subseteq Z.d \sqsubseteq \bigsqcup Z_0 \ \& \ Z_0 \text{ is finite.} \end{aligned}$$

($\Leftarrow$) Suppose $d \sqsubseteq \bigsqcup Z$ with $Z \subseteq \mathcal{D}$ directed. By the condition $d \sqsubseteq \bigsqcup Z_0$ for some finite $Z_0 \subseteq Z$. If Z_0 is non-empty, then by the directedness of Z there exists a $z \in Z$ such that $z \sqsupseteq \bigsqcup Z_0 \sqsupseteq d$. If Z_0 is empty, then $d = \bot$ and we can take an arbitrary element z in the non-empty Z satisfying $d \sqsubseteq z$. ∎

[1]In general it is not true that if $d \sqsubseteq e \in \mathcal{K}(\mathcal{D})$, then $d \in \mathcal{K}(\mathcal{D})$; take for example $\omega + 1$ in the ordinal $\omega + \omega = \{0, 1, 2, \ldots \omega, \omega + 1, \omega + 2, \ldots\}$.

16.1.8. DEFINITION. (i) Let $\mathcal{D},\mathcal{E}$ be complete lattices. Then $f:\mathcal{D}\to\mathcal{E}$ is called *(Scott) continuous* iff for all directed $X\subseteq\mathcal{D}$ one has

$$f(\bigsqcup X)=\bigsqcup f(X)\ (=\bigsqcup\{f(x)\mid x\in X\}).$$

(ii) $[\mathcal{D}\to\mathcal{E}]=\{f:\mathcal{D}\to\mathcal{E}\mid f$ is Scott continuous functions$\}$.

16.1.9. PROPOSITION. *Let $\mathcal{D},\mathcal{E}$ be algebraic lattices.*

(i) *Let $f\in[\mathcal{D}\to\mathcal{E}]$. Then for $x\in\mathcal{D}$*

$$f(x)=\bigsqcup\{f(e)\mid e\sqsubseteq x\ \&\ e\in\mathcal{K}(\mathcal{D})\}.$$

(ii) *Let $f,g\in[\mathcal{D}\to\mathcal{E}]$. Suppose $f\restriction\mathcal{K}(\mathcal{D})=g\restriction\mathcal{K}(\mathcal{D})$. Then $f=g$.*

PROOF. (i) Use that $x=\bigsqcup\{e\mid e\sqsubseteq x\}$ is a directed sup and that f is continuous.

(ii) By (i). ∎

16.1.10. DEFINITION. The category **ALG** has as objects the ω-algebraic complete lattices and as morphisms the Scott continuous functions.

The main result of this section is Corollary 16.1.19 stating the equivalence between the categories **MSL** and **ALG**. First we define the functors establishing this equivalence.

16.1.11. DEFINITION. (i) Given an $\mathcal{S}\in\mathbf{MSL}$, a *filter* over $\mathcal{S}$ is a non-empty set $X\subseteq S$ such that the following hold.

(1) $s\in X\ \&\ s\leq s'\ \Rightarrow\ s'\in X$;

(2) $s,s'\in X\ \Rightarrow\ s\cap s'\in X$.

(ii) Write $\mathcal{F}^S=\{X\subseteq S\mid X$ is a filter over $S\}$.

16.1.12. DEFINITION. Let $\mathcal{S}\in\mathbf{MSL}$.

(i) For $s\in S$ write $\uparrow s=\{x\in S\mid s\sqsubseteq x\}$.

(ii) For $X\subseteq S$ define $\uparrow X$ to be the least filter over S containing X; it can be described explicitly by

$$\uparrow X=\{x\in S\mid \exists X_0\subseteq X.X_0 \text{ is finite and } x\geq\bigcap X_0\}.$$

Note that $\uparrow\emptyset=\{\top\}$, since $\bigcap\emptyset=\top$.

16.1.13. PROPOSITION. *Let $S\in\mathbf{MSL}$.*

(i) *$\mathcal{F}^S=\langle\mathcal{F}^S,\subseteq\rangle$ is a complete lattice, with for $\mathcal{X}\subseteq\mathcal{F}^S$ the sup is*

$$\bigsqcup\mathcal{X}=\uparrow(\cup\mathcal{X}).$$

(ii) *For $s\in S$ one has $\uparrow s=\uparrow\{s\}$ and $\uparrow s\in\mathcal{F}^S$.*

(iii) *For $s,s'\in S$ one has $\uparrow s\sqcup\uparrow s'=\uparrow(s\cap s')$.*

(iv) *For* $X \in \mathcal{F}^S$ *one has*

$$\begin{array}{rcl} X & = & \bigsqcup\{\uparrow s \mid s \in X\} \\ & = & \bigsqcup\{\uparrow s \mid \uparrow s \subseteq X\} \\ & = & \bigcup\{\uparrow s \mid s \in X\} \\ & = & \bigcup\{\uparrow s \mid \uparrow s \subseteq X\}. \end{array}$$

(v) $\{\uparrow s \mid s \in S\}$ *is the set of finite elements of* $\mathcal{F}^S$.

(vi) $\mathcal{F}^S \in \mathbf{ALG}$.

PROOF. Easy. ■

16.1.14. DEFINITION. The functor $\mathsf{Flt} : \mathbf{MSL} \to \mathbf{ALG}$ is defined as follows.

$$\mathsf{Flt}(S) = \langle \mathcal{F}^S, \subseteq \rangle.$$

For an arrow $\mu : S \to S'$ we define $\mathsf{Flt}(\mu) : \mathcal{F}^S \to \mathcal{F}^{S'}$ by

$$\mathsf{Flt}(\mu)(X) = \mu(X) = \{s' \mid \exists s \in X . s\ \mu\ s'\}.$$

16.1.15. LEMMA. *Let* $\mathcal{D} \in \mathbf{ALG}$. *Define* $\mathcal{K}(\mathcal{D}) = \{d \in \mathcal{D} \mid d$ *is compact*$\}$. *On* $\mathcal{K}(\mathcal{D})$ *define*

$$d \leq e \ \Leftrightarrow \ e \sqsubseteq d.$$

Then

$$\begin{array}{rcl} d \cap_{\leq} e & = & d \sqcup_{\sqsubseteq} e; \\ \bot_{\leq} & = & \top_{\sqsubseteq}; \\ \top_{\leq} & = & \bot_{\sqsubseteq}. \end{array}$$

Hence $(\mathcal{K}(\mathcal{D}), \leq, \cap_{\leq}, \top_{\leq}) \in \mathbf{MSL}$.

PROOF. Immediate. ■

16.1.16. DEFINITION. We define a functor $\mathsf{Cmp} : \mathbf{ALG} \to \mathbf{MSL}$ as follows.

(i) On objects $\mathcal{D} \in \mathbf{ALG}$ one defines Cmp as follows.

$$\mathsf{Cmp}((D, \sqsubseteq)) = (\mathcal{K}(D), \leq, \cap_{\leq}, \top_{\leq}).$$

(ii) On arrows $f \in \mathbf{ALG}(D, D')$ one defines the approximable map $\mathtt{Cmp}(f)$

$$c\ \mathsf{Cmp}(f)\ c' \ \Leftrightarrow \ c' \sqsubseteq f(c),$$

so that $\mathsf{Cmp}(f) \in \mathbf{MSL}(\mathsf{Cmp}(\mathcal{D}), \mathsf{Cmp}(\mathcal{D}'))$.

16.1.17. PROPOSITION. (i) *Let* $S \in \mathbf{MSL}$. *Then* $\varphi : S \to \mathcal{K}(\mathcal{F}^S)$ *defined by* $\varphi(s) = \uparrow s$ *is an* $\mathbf{MSL}_0$ *isomorphism. Hence also in* $\mathbf{MSL}$ *one has* $S \cong \mathcal{K}(\mathcal{F}^S)$.

$$\begin{tikzcd} \mathcal{S} \arrow[rr, dotted, "\mathsf{Flt}"] \arrow[dr, "\uparrow"'] & & \mathcal{F}^{\mathcal{S}} \arrow[dl, dotted, "\mathsf{Cmp}"] \\ & \mathcal{K}(\mathcal{F}^{\mathcal{S}}) & \end{tikzcd}$$

(ii) *Let* $D\in\mathbf{ALG}$. *Then* $m : \mathcal{F}^{\mathcal{K}(D)}\to D$ *defined by* $m(X) = \bigsqcup X$, *is an* $\mathbf{ALG}$ *isomorphism with inverse* $n : D\to\mathcal{F}^{\mathcal{K}(D)}$ *defined by* $n(d) = \{c\in\mathcal{K}(D) \mid c \sqsubseteq d\}$.

$$\begin{tikzcd}
\mathcal{K}(\mathcal{D}) \arrow[dr, dotted, "\mathsf{Flt}"'] & & \mathcal{D} \arrow[ll, dotted, "\mathsf{Cmp}"'] \\
& \mathcal{F}^{\mathcal{K}(\mathcal{D})} \arrow[ur, "\bigsqcup"'] &
\end{tikzcd}$$

PROOF. (i) By proposition 16.1.13(v) φ is a surjection. It is also 1-1, since $\uparrow s = \uparrow t \ \Rightarrow\ s \le t \le s \ \Rightarrow\ s = t$. Moreover, φ preserves $\le$:

$$\begin{array}{rcll}
s \le t & \Leftrightarrow & \uparrow t \subseteq \uparrow s & \\
 & \Leftrightarrow & \uparrow s \le \uparrow t, & \text{by definition of } \le \text{ on } \mathcal{K}(\mathcal{F}^{\mathcal{S}}).
\end{array}$$

Also φ preserves $\cap$:

$$\begin{array}{rcll}
\varphi(s\cap t) & = & \uparrow(s\cap t) & \\
 & = & \uparrow s \sqcup \uparrow t & \text{in } \mathcal{F}^{\mathcal{S}}, \text{ by 16.1.13(iii)}, \\
 & = & \uparrow s \cap \uparrow t & \text{in } \mathcal{K}(\mathcal{F}^{\mathcal{S}}), \text{ by definition of } \le \text{ on } \mathcal{K}(\mathcal{F}^{\mathcal{S}}), \\
 & = & \varphi(s)\cap\varphi(t). &
\end{array}$$

Finally $\varphi(\top_{\mathcal{S}}) = \uparrow\top_{\mathcal{S}} = \top_{\mathcal{K}(\mathcal{F}^{\mathcal{S}})}$, since $\{\top_{\mathcal{S}}\}$ is the $\subseteq$-smallest filter; hence φ preserves $\top$.

Similarly one can show that φ^{-1} preserves $\le, \sqcap_{\le}$ and $\top_{\le}$, since by lemma 16.1.13(v) the elements of $\mathcal{K}(\mathcal{F}^{\mathcal{S}})$ are of the form $\uparrow s$ with $s\in\mathcal{S}$ and $\varphi^{-1}(\uparrow s) = s$.

Therefore $\mathcal{S} \cong \mathcal{K}(\mathcal{F}^{\mathcal{S}})$ in $\mathbf{MSL}_0$, hence in $\mathbf{MSL}$, by lemma 16.1.5.

(ii) We have $m\circ n = \mathbf{1}_{\mathcal{D}}$ and $n\circ m = \mathbf{1}_{\mathcal{F}^{\mathcal{K}(\mathcal{D})}}$:

$$\begin{array}{rcll}
m(n(d)) & = & \sqcup\{c\in\mathcal{K}(\mathcal{D}) \mid c\sqsubseteq d\} & \\
 & = & d, & \text{since } \mathcal{D}\in\mathbf{ALG}. \\
n(m(X)) & = & \{c\in\mathcal{K}(\mathcal{D}) \mid c\sqsubseteq \sqcup X\} & \\
 & = & \{c\in\mathcal{K}(\mathcal{D}) \mid c\in X\}, & \text{since one has} \\
c\sqsubseteq_{\mathcal{D}} \sqcup X & \Leftrightarrow & \exists x\in X\ c\sqsubseteq_{\mathcal{D}} x, & \text{since } c\in\mathcal{K}(\mathcal{D}) \text{ and } X\subseteq\mathcal{K}(\mathcal{D})\subseteq\mathcal{D} \text{ is a} \\
 & & & \text{filter w.r.t. } \le \text{ hence directed w.r.t. } \sqsubseteq \\
 & \Leftrightarrow & \exists x\in X\ x\le_{\mathcal{K}(\mathcal{D})} c & \\
 & \Leftrightarrow & c\in X, & \text{as } X \text{ is a filter on } \mathcal{K}(\mathcal{D}).
\end{array}$$

Moreover, n is Scott continuous. Indeed, let $Z\subseteq\mathcal{D}$ be directed; then

$$\begin{array}{rcll}
n(\bigsqcup_{d\in Z} d) & = & \{c\in\mathcal{K}(\mathcal{D}) \mid c\sqsubseteq \bigsqcup_{d\in Z} d\} & \\
 & = & \{c\in\mathcal{K}(\mathcal{D}) \mid \exists d\in Z\ c\sqsubseteq d\}, & \text{since } Z \text{ is directed and } c\in\mathcal{K}(\mathcal{D}), \\
 & = & \bigcup_{d\in Z}\{c\in\mathcal{K}(\mathcal{D}) \mid c\sqsubseteq d\} & \\
 & = & \bigcup_{d\in Z} n(d). &
\end{array}$$

A simpler reasoning shows that m preserves even arbitrary sups. ■

16.1.18. Corollary. (i) *Let* $\mathcal{S}\in\mathbf{MSL}$. *Then*

$$\mathcal{S} \cong \mathcal{K}(\mathcal{F}^{\mathcal{S}}).$$

(ii) *Let* $\mathcal{D}\in\mathbf{ALG}$. *Then*

$$\mathcal{D} \cong \mathcal{F}^{\mathcal{K}(\mathcal{D})}.$$

16.1.19. Corollary. *The categories* $\mathbf{MSL}$ *and* $\mathbf{ALG}$ *are equivalent; in fact the isomorphisms in 16.1.17 form natural isomorphisms showing* $\mathsf{Cmp} \circ \mathsf{Flt} \cong \mathbf{1}_{\mathbf{MSL}}$ *and* $\mathsf{Flt} \circ \mathsf{Cmp} \cong \mathbf{1}_{\mathbf{ALG}}$.

Proof. First one has to show that in $\mathbf{MSL}$ the following diagram commutes.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\mu_\uparrow"] \arrow[d, "\mu"'] & \mathcal{K}(\mathcal{F}^{\mathcal{S}}) \arrow[d, "\mathsf{Cmp}(\mathsf{Flt}(\mu))"] \\
\mathcal{S}' \arrow[r, "\mu_\uparrow"] & \mathcal{K}(\mathcal{F}^{\mathcal{S}'})
\end{tikzcd}$$

Notice that for $X\in\mathcal{K}(\mathcal{F}^{\mathcal{S}}), X'\in\mathcal{K}(\mathcal{F}^{\mathcal{S}'})$

$$\begin{aligned}
X \ \mathsf{Cmp}(\mathsf{Flt}(\mu))\ X' &\Leftrightarrow X' \subseteq \mathsf{Flt}(X)(\mu) \\
&\Leftrightarrow X' \subseteq \mu(X).
\end{aligned}$$

Moreover X, X' can be written as $X = {\uparrow}t, X' = {\uparrow}t'$, with $t\in\mathcal{S}, t'\in\mathcal{S}$. Now for arbitrary $s\in\mathcal{S}, {\uparrow}t'\in\mathcal{K}(\mathcal{F}^{\mathcal{S}'})$ one has along one side

$$\begin{aligned}
s \ \mathsf{Cmp}(\mathsf{Flt}(\mu)) \circ \mu_\uparrow \ {\uparrow}t' &\Leftrightarrow s\ \mu_\uparrow \circ_r \mathsf{Cmp}(\mathsf{Flt}(\mu))\ {\uparrow}t' \\
&\Leftrightarrow \exists t.\ s\ \mu_\uparrow\ {\uparrow}t\ \mathsf{Cmp}(\mathsf{Flt}(\mu))\ {\uparrow}t' \\
&\Leftrightarrow \exists t.\ {\uparrow}s \leq {\uparrow}t\ \&\ {\uparrow}t' \subseteq \mu({\uparrow}t) \\
&\Leftrightarrow \exists t, u.\ s \leq t\ \&\ u\ \mu\ t' \\
&\Leftrightarrow s\ \mu\ t'.
\end{aligned}$$

Going the other side one has

$$\begin{aligned}
s\ \mu_\uparrow \circ \mu\ {\uparrow}t' &\Leftrightarrow s\ \mu \circ_r \mu_\uparrow\ {\uparrow}t' \\
&\Leftrightarrow \exists s'.\, s\ \mu\ s'\ \mu_\uparrow\ {\uparrow}t' \\
&\Leftrightarrow \exists s'.\, s\ \mu\ s'\ \&\ {\uparrow}s' \leq {\uparrow}t' \\
&\Leftrightarrow \exists s'.\, s\ \mu\ s'\ \&\ s' \leq t' \\
&\Leftrightarrow s\ \mu\ t'.
\end{aligned}$$

Hence the diagram commutes. This suffices to conclude $\mathsf{Cmp} \circ \mathsf{Flt} \cong \mathbf{1}_{\mathbf{MSL}}$ (by using the inverse morphisms).

Secondly one has to show that in **ALG** the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}^{\mathcal{K}(\mathcal{D})} \arrow[r, "\bigsqcup"] \arrow[d, "\mathsf{Flt}(\mathsf{Cmp}(f))"'] & \mathcal{D} \arrow[d, "f"] \\
\mathcal{F}^{\mathcal{K}(\mathcal{D}')} \arrow[r, "\bigsqcup"] & \mathcal{D}'
\end{tikzcd}$$

Now for $X\in\mathcal{F}^{\mathcal{K}(\mathcal{D})}$ one has

$$\begin{aligned}
\mathsf{Flt}(\mathsf{Cmp}(f))(X) &= \mathsf{Cmp}(f)(X) \\
&= \{d'\in\mathcal{K}(\mathcal{D}') \mid \exists d\in X.\, d\ \mathsf{Cmp}(f)\ d'\} \\
&= \{d'\in\mathcal{K}(\mathcal{D}') \mid \exists d\in X.\, d' \sqsubseteq f(d)\}.
\end{aligned}$$

Hence, using also the continuity of f,

$$\bigsqcup(\mathsf{Flt}(\mathsf{Cmp}(f))(X)) = \bigsqcup_{d\in X} f(d) = f(\bigsqcup X),$$

and the diagram commutes. As before we have $\mathsf{Flt}\circ\mathsf{Cmp} \cong \mathbf{1}_{\mathbf{ALG}}$. ■

This result is a special case of Stone duality, cf. Johnstone [1986] (II, 3.3).

16.2. Natural type structures and lambda structures 26.2.2004:1018

Inspired by previous section we now enrich the objects of **MSL** and **ALG**.

An intersection type structure with top is just an object of **MSL** with an additional operation $\to$ on its domain

$$\mathcal{S} = \langle S, \leq, \cap, \top, \to\rangle.$$

Natural type structures $\mathcal{M}$ were defined in definition 14.1.13: they are intersection type structures with element Ω and satisfy (Ω), $(\Omega\eta)$, $(\to\cap)$ and (η).

16.2.1. Definition. We will define a category **NTS**.

(i) The objects of **NTS** are the natural type structures.

(ii) The morphisms of **NTS** are approximable mappings $\mu\in\mathbf{MSL}(\mathcal{S},\mathcal{S}')$ which satisfy the following condition. For all $s_1, s_2\in S$, $t'\in S'$,

$$(\dagger)\quad \exists s'\in S'.[s_2\,\mu\, s' \text{ and } s_1\,\mu\,(s'\to t')] \;\Leftrightarrow\; \exists t\in S.[s_1 \leq s_2\to t \text{ and } t\,\mu\, t'].$$

Notice that the condition above specializes to $\mathbf{MSL}_0$ morphisms as follows: if $\mu = \mu_f$, with $f : S\to S'$, then $\mu_f\in\mathbf{NTS}(\mathcal{S},\mathcal{S}')$ iff for all $s_1, s_2\in S$, $t'\in S'$,

$$(\ddagger)\quad f(s_1) \leq f(s_2)\to t' \;\Leftrightarrow\; \exists t\in S. s_1 \leq s_2\to t \text{ and } f(t) \leq t'.$$

16.2.2. REMARK. Let $\nabla \in \{\text{Scott}, \text{Park}, \text{CDZ}, \text{HR}, \text{DHM}, \text{BCD}\}$. Then we have $\mathcal{T}^{\nabla} \in \mathbf{NTS}$. But $(\Omega\eta)$ does not hold for $\mathcal{T}^{\mathrm{AO}}$, see exercise 15.4.2(ii), hence $\mathcal{T}^{\mathrm{AO}} \notin \mathbf{NTS}$. In Definition 14.1.13 we defined the notion of *weak* natural type structure, replacing $(\Omega\eta)$ by (Ω-lazy). Clearly $\mathcal{T}^{\mathrm{AO}}$ is a weak natural type structure. These structures will be considered in Exercises 16.6.1-16.6.3.

16.2.3. PROPOSITION. *Let $\mu \in \mathbf{MSL}(\mathcal{S}, \mathcal{S}')$. Then μ is an* **NTS**-*isomorphism if and only if $\mu = \mu_f$, with $f : S \to S'$ monotone bijection with inverse f^{-1} monotone, and such that, for each $s_1, s_2, t \in S$, we have:*

$$(*) \quad s_1 \leq s_2 \to t \;\Leftrightarrow\; f(s_1) \leq f(s_2) \to f(t).$$

PROOF. $(\Rightarrow)$ This follows immediately by Lemma 16.1.4 and the fact that condition $(\ddagger)$ implies $(*)$ (use the monotonicity of f^{-1}). $(\Leftarrow)$ It sufficies to prove that condition $(\ddagger)$ holds for f and f^{-1}. We have

$$\begin{array}{rcll} \exists t. s_1 \leq s_2 \to t \;\&\; f(t) \leq t' & \Leftrightarrow & \exists t. f(s_1) \leq f(s_2) \to f(t) \;\&\; f(t) \leq t' & \text{by } (*) \\ & \Leftrightarrow & f(s_1) \leq f(s_2) \to t' & \text{by } (\eta). \end{array}$$

So $(\ddagger)$ holds for f. Since $(*)$ implies

$$s_1' \leq s_2' \to t' \;\Leftrightarrow\; f^{-1}(s_1') \leq f^{-1}(s_2') \to f^{-1}(t'),$$

the same steps lead to the proof that $(\ddagger)$ holds for f^{-1}. ■

16.2.4. PROPOSITION. *$\mathcal{S}$ is a natural type structure if and only if for each $r \in S$, the relation $\mu_r \subseteq S \times S$ defined by*

$$s_1 \, \mu_r \, t_1 \;\Leftrightarrow\; r \leq s_1 \to t_1,$$

is an approximable mapping.

PROOF. $(\Rightarrow)$ Suppose that $\mathcal{S}$ is natural. We prove that for any $r \in S$, μ_r is an approximable mapping. First of all, since by (Ω-η), $r \leq \Omega \to \Omega$, we have $\Omega \, \mu_r \, \Omega$. Suppose

$$s_2 \leq s_1 \, \mu_r \, t_1 \leq t_2.$$

Since by definition of μ_r we have $r \leq s_1 \to t_1$, and by (η), $s_1 \to t_1 \leq s_2 \to t_2$, we get $r \leq s_2 \to t_2$, that is $s_2 \, \mu_r t_2$. Finally, suppose

$$s_1 \, \mu_r \, t_1 \text{ and } s_1 \, \mu_r \, t_2.$$

Then, by ($\to$-$\cap$), we get $r \leq s_1 \to (t_1 \cap t_2)$, hence $s_1 \, \mu_r (t_1 \cap t_2)$. This completes the proof of the $(\Rightarrow)$ case.

$(\Leftarrow)$ Suppose that for any $r \in S$, μ_r is an approximable mapping. In particular so is μ_Ω, hence it holds $\Omega \, \mu_\Omega \, \Omega$, which implies ($\Omega$-$\eta$). Let $s_2 \leq s_1$ and $t_1 \leq t_2$. Since $s_1 \to t_1 \leq s_1 \to t_1$, we have $s_1 \, \mu_{s_1 \to t_1} \, t_1$. Since $\mu_{s_1 \to t_1}$ is an approximable mapping, it follows $s_2 \, \mu_{s_1 \to t_1} \, t_2$, so $s_1 \to t_1 \leq s_2 \to t_2$, hence (η) holds. Finally, let $r = (s \to t_1) \cap (s \to t_2)$. Then, since $r \leq s \to t_1$ and $r \leq s \to t_2$, we have $s \, \mu_r \, t_1$ and $s \, \mu_r \, t_2$, hence $s \, \mu_r \, (t_1 \cap t_2)$, which implies ($\to$-$\cap$).

The following definitions make sense for more general structures, but we consider them for **ALG**.

16.2.5. Definition. (i) If $\mathcal{D}\in\mathbf{ALG}$, then $[\mathcal{D}\to\mathcal{D}]$ denotes the set of continuous functions from $\mathcal{D}$ to $\mathcal{D}$.

(ii) $[\mathcal{D}\to\mathcal{D}]$ is partially ordered pointwise:

$$f \sqsubseteq g \ \Leftrightarrow\ \forall x\in\mathcal{D}.f(x) \sqsubseteq g(x).$$

(iii) If $a,b\in\mathcal{D}$, then $a \Rightarrow b$ is the *step function* defined by

$$\begin{array}{rcll}(a \Rightarrow b)(d) & = & b, & \text{if } a \sqsubseteq d;\\ & = & \bot, & \text{else.}\end{array}$$

16.2.6. Lemma. *$[\mathcal{D}\to\mathcal{D}]$ is a complete lattice with*

$$(\bigsqcup_{f\in X} f)(d) = \bigsqcup_{f\in X} f(d).$$

16.2.7. Lemma. *For $a,b,a_1,a_2,b_1,b_2\in\mathcal{D}$ and $f\in[\mathcal{D}\to\mathcal{D}]$ one has*

(i) *a compact $\ \Rightarrow\ a \Rightarrow b$ is continuous.*

(ii) *$a \Rightarrow b$ is continuous and $b \neq \bot\ \Rightarrow\ a$ is compact.*

(iii) *b compact $\ \Leftrightarrow\ a \Rightarrow b$ compact.*

(iv) $b \sqsubseteq f(a)\ \Leftrightarrow\ (a \Rightarrow b) \sqsubseteq f$.

(v) $a_2 \sqsubseteq a_1\ \&\ b_1 \sqsubseteq b_2\ \Rightarrow\ (a_1 \Rightarrow b_1) \sqsubseteq (a_2 \Rightarrow b_2)$.

(vi) $(a_1 \Rightarrow b_1) \sqcup (a_2 \Rightarrow b_2) \sqsubseteq (a_1 \sqcap a_2) \Rightarrow (b_1 \sqcup b_2)$.

Proof. Easy. ■

16.2.8. Lemma. *For $c,d,a_1,b_1,\dots,a_n,b_n\in\mathcal{D}$ and $f\in[\mathcal{D}\to\mathcal{D}]$ one has*

$$\begin{array}{l}(c \Rightarrow d) \sqsubseteq (a_1 \Rightarrow b_1) \sqcup \ldots \sqcup (a_n \Rightarrow b_n)\ \Leftrightarrow\\ \qquad\qquad \Leftrightarrow\ \exists I\subseteq\{1,\dots,n\}\,[\sqcup_{i\in I}a_i \sqsubseteq c\ \&\ d \sqsubseteq \sqcup_{i\in I}b_i].\end{array}$$

Clearly in $(\Rightarrow)$ we have $I \neq \emptyset$ if $d \neq \bot$.

Proof. Easy. ■

16.2.9. Proposition. *Let $\mathcal{D}\in\mathbf{ALG}$.*

(i) *For $f\in[\mathcal{D}\to\mathcal{D}]$ one has $f = \bigsqcup\{a \Rightarrow b \mid b \sqsubseteq f(a),\ a,b$ compact in $\mathcal{D}\}$.*

(ii) $[\mathcal{D}\to\mathcal{D}]\in\mathbf{ALG}$.

Proof. (i) It suffices to show that RHS and LHS are equal when applied to an arbitrary element $d\in\mathcal{D}$.

$$\begin{array}{rcl}f(d) & = & f(\bigsqcup\{a \mid a \sqsubseteq d\ \&\ a\in\mathcal{K}(\mathcal{D})\})\\ & = & \bigsqcup\{f(a) \mid a \sqsubseteq d\ \&\ a\in\mathcal{K}(\mathcal{D})\})\\ & = & \bigsqcup\{b \mid b \sqsubseteq f(a)\ \&\ a \sqsubseteq d\ \&\ a,b\in\mathcal{K}(\mathcal{D})\})\\ & = & \bigsqcup\{a \Rightarrow b(d) \mid b \sqsubseteq f(a)\ \&\ a \sqsubseteq d\ \&\ a,b\in\mathcal{K}(\mathcal{D})\})\\ & = & \bigsqcup\{a \Rightarrow b(d) \mid b \sqsubseteq f(a)\ \&\ a,b\in\mathcal{K}(\mathcal{D})\})\\ & = & (\bigsqcup\{a \Rightarrow b(\mid b \sqsubseteq f(a)\ \&\ a,b\in\mathcal{K}(\mathcal{D})\}))(d).\end{array}$$

(ii) It remains to show that there are only countably many compact elements in $[\mathcal{D}{\to}\mathcal{D}]$. Let f be compact. By (iii) $f = \bigsqcup\{a \Rightarrow b \mid b \sqsubseteq f(a) \ \&\ a, b{\in}\mathcal{K}(\mathcal{D})\}$, so we have by lemma 16.1.7 for some $a_1, \dots, a_n, b_1, \dots, b_n{\in}\mathcal{K}(\mathcal{D})$ that

$$f = (a_1 \Rightarrow b_1) \sqcup \ldots \sqcup (a_n \Rightarrow b_n). \qquad (*)$$

Since $\mathcal{K}(\mathcal{D})$ is countable, there are only countably many expressions in the RHS of (*). (The cardinality is $\Sigma_n n.\aleph_0^2 = \aleph_0$.) Therefore ther are countable many compact $f{\in}[\mathcal{D}{\to}\mathcal{D}]$. (There may be more expressions on the RHS for one f, but this results in less compact elements.) ∎

16.2.10. Definition. (i) A *λ-structure* is a triple $\langle\mathcal{D}, F, G\rangle$, where $\mathcal{D}$ is in **ALG** and $F : \mathcal{D}{\to}[\mathcal{D}{\to}\mathcal{D}]$, $G : [\mathcal{D}{\to}\mathcal{D}]{\to}\mathcal{D}$ are Scott continuous functions.

(ii) In a λ-structure one defines

$$d \cdot e \;=\; F(d)(e).$$

(iii) A *natural* λ-structure is one that satisfies

$$\begin{array}{lcl} F \circ G & \sqsupseteq & \mathbf{1}_{\mathcal{D}\to\mathcal{D}} \\ G \circ F & \sqsubseteq & \mathbf{1}_{\mathcal{D}} \end{array}$$

(iv) The category **NLS** has as objects natural lambda structures and as morphisms Scott continuous maps m satisfying

$$m(d \cdot e) \;=\; m(d) \cdot m(e).$$

There is no need for a condition on m in terms of G, as we will see that F completely determines G.

16.2.11. Lemma. *Let $\mathcal{D}{\in}$**NLS**.*

(i) *For $f{\in}[\mathcal{D}{\to}\mathcal{D}]$*

$$f \text{ is compact } \Rightarrow G(f) \text{ is compact.}$$

(ii) *For $f_1, f_2{\in}[\mathcal{D}{\to}\mathcal{D}]$*

$$G(f_1 \sqcup f_2) = G(f_1) \sqcup G(f_2).$$

(iii) *F determines G: for all $f{\in}[\mathcal{D}{\to}\mathcal{D}]$*

$$G(f) = \sqcap\{d \mid f \sqsubseteq F(d)\}.$$

Proof. (i) Suppose f is compact. Then

$$\begin{array}{rlll} G(f) \sqsubseteq \bigsqcup_{z\in Z} z & \Rightarrow & F(G(f)) \sqsubseteq \bigsqcup_{z\in Z} F(z), & \text{since } F \text{ is continuous,} \\ & \Rightarrow & f \sqsubseteq \bigsqcup_{z\in Z} F(z), & \text{since } \mathcal{D} \text{ is natural,} \\ & \Rightarrow & \exists z{\in}Z.f \sqsubseteq F(z), & \text{since } f \text{ is compact} \\ & & & \text{and } F(Z) \text{ is directed,} \\ & \Rightarrow & \exists z{\in}Z.G(f) \sqsubseteq G(F(z)), & \text{since } G \text{ is monotonic,} \\ & \Rightarrow & \exists z{\in}Z.G(f) \sqsubseteq z, & \text{since } \mathcal{D} \text{ is natural.} \end{array}$$

(ii) Since $G(f_i) \sqsubseteq G(f_1 \sqcup f_2)$, by monotonicity of G, one has $G(f_1) \sqcup G(f_2) \sqsubseteq G(f_1 \sqcup f_2)$. Conversely,

$$\begin{array}{rcll} G(f_1 \sqcup f_2) & \sqsubseteq & G[F(G(f_1)) \sqcup F(G(f_2))], & \text{since } F \circ G \sqsupseteq \mathbf{1}_{[\mathcal{D}\to\mathcal{D}]}, \\ & \sqsubseteq & G(F[G(f_1) \sqcup G(f_2)]), & \text{by monotonicity of } F, \\ & \sqsubseteq & G(f_1) \sqcup G(f_2), & \text{since } G \circ F \sqsubseteq \mathbf{1}_{\mathcal{D}}. \end{array}$$

(iii) It suffices to show that $G(f) \sqsubseteq d \Leftrightarrow f \sqsubseteq F(d)$. Indeed,

$$\begin{array}{rcll} G(f) \sqsubseteq d & \Rightarrow & F(G(f)) \sqsubseteq F(d), & \text{since } F \text{ is monotonic} \\ & \Rightarrow & f \sqsubseteq F(d), & \text{as } \mathbf{1}_{[\mathcal{D}\to\mathcal{D}]} \sqsubseteq F \circ G, \\ & \Rightarrow & G(f) \sqsubseteq G(F(d)), & \text{since } G \text{ is monotonic,} \\ & \Rightarrow & G(f) \sqsubseteq d & \text{as } G \circ F \sqsubseteq \mathbf{1}_{\mathcal{D}}. \blacksquare \end{array}$$

Now we can give the crucial definition.

16.2.12. Definition (Filter structure). Let $\mathcal{S}$ be an object of **NTS**.

(i) On $\mathcal{F}^{\mathcal{S}} \in \mathbf{ALG}$ we define $F^{\mathcal{S}} : \mathcal{F}^{\mathcal{S}} \to [\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}]$ and $G^{\mathcal{S}} : [\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}] \to \mathcal{F}^{\mathcal{S}}$ as follows.

$$\begin{array}{rcl} F^{\mathcal{S}}(X)(Y) & = & \{b \in \mathcal{S} \mid \exists a \in Y. a \to b \in X\} \\ G^{\mathcal{S}}(f) & = & \uparrow\{a \to b \in \mathcal{S} \mid b \in f(\uparrow a)\} \end{array}$$

By (η), $(\Omega\text{-}\eta)$ and $(\to\text{-}\cap)$ it follows that $F^{\mathcal{S}}(X)(Y)$ is actually a filter.

(ii) The triple $\mathcal{F}^{\mathcal{S}} = \langle \mathcal{F}^{\mathcal{S}}, F^{\mathcal{S}}, G^{\mathcal{S}} \rangle$ is called the *filter structure* over $\mathcal{S}$.

The proof of the following lemma follows easily from the definition of **NTS**.

16.2.13. Lemma. *Let $\mathcal{S}$ be an object of* **NTS** *and $s_1, s_2 \in S$. Then*

(i) $F^{\mathcal{S}}(\uparrow s_1)(\uparrow s_2) = \{t \in S \mid s_1 \leq s_2 \to t\}$.

(ii) $G^{\mathcal{S}}(\uparrow s_1 \Rightarrow \uparrow s_2) = \uparrow(s_1 \to s_2)$.

In the following we show that the functor Flt restricts to a functor from **NTS** to **NLS**.

16.2.14. Proposition. *Let $\mathcal{S}$ be in* **NTS**. *Then $\mathcal{F}^{\mathcal{S}}$ is in* **NLS**.

Proof. The continuity of $F^{\mathcal{S}}$ and $G^{\mathcal{S}}$ follows immediately since the sup of directed sets of filters is their union.

It remains to show that $\mathcal{F}^{\mathcal{S}}$ is natural. Now for $f \in [\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}]$ and $X \in \mathcal{F}^{\mathcal{S}}$

$$\begin{array}{rcll} F^{\mathcal{S}}(G^{\mathcal{S}}(f))(X) & = & \{b \mid \exists a \in X. a \to b \in G^{\mathcal{S}}(f)\}, & \text{by definition of } F^{\mathcal{S}}, \\ & \supseteq & \{b \mid \exists a \in X. b \in f(\uparrow a)\}, & \text{use the definition of } G^{\mathcal{S}}, \\ & = & \bigcup_{a \in X} \{b \mid b \in f(\uparrow a)\} & \\ & = & \bigcup_{a \in X} f(\uparrow a) & \\ & = & f(\bigcup_{a \in X} \uparrow a), & \text{by continuity of } f, \\ & = & f(X), & \text{since } X \text{ is a filter.} \end{array}$$

Therefore $F^{\mathcal{S}}(G^{\mathcal{S}}(f)) \sqsupseteq f$, hence $F^{\mathcal{S}} \circ G^{\mathcal{S}} \sqsupseteq \mathbf{1}_{[\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}]}$.

Moreover, for $X \in \mathcal{F}^{\mathcal{S}}$

$$\begin{array}{rcll} G^{\mathcal{S}}(F^{\mathcal{S}}(X)) & = & \uparrow\{a\to b \mid b\in F^{\mathcal{S}}(X)(\uparrow a)\}, & \text{by definition of } G^{\mathcal{S}}, \\ & = & \uparrow\{a\to b \mid \exists a'\in\uparrow a. a'\to b\in X\}, & \text{by definition of } F^{\mathcal{S}}, \\ & = & \uparrow\{a\to b \mid a\to b\in X\}, & \text{using } (\eta) \text{ and that} \\ & & & X \text{ is a filter}, \\ & \subseteq & X, & \text{since } X \text{ is a filter.} \end{array}$$

Therefore $G^{\mathcal{S}} \circ F^{\mathcal{S}} \subseteq \mathbf{1}_{\mathcal{F}^{\mathcal{S}}}$. ∎

16.2.15. Proposition. *Let* $\mu \in \mathbf{NTS}(\mathcal{S}, \mathcal{S}')$. *Then* $\mathsf{Flt}(\mu) \in \mathbf{NLS}(\mathcal{F}^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}'})$.

Proof. Let $m = \mathsf{Flt}(\mu)$. Recall that μ satisfies condition (†) of Definition 16.2.1. We have to prove that for any filters $X, Y \in \mathcal{F}^{\mathcal{S}}$,

$$m(F^{\mathcal{S}}(X)(Y)) = F^{\mathcal{S}'}(m(X))(m(Y))$$

Since m and $F^{\mathcal{S}}$ are continuous, it is sufficient to prove that for any $s_1, s_2 \in S$, it holds

$$m(F^{\mathcal{S}}(\uparrow s_1)(\uparrow s_2)) = F^{\mathcal{S}'}(m(\uparrow s_1))(m(\uparrow s_2))$$

We have

$$\begin{array}{rcll} m(F^{\mathcal{S}}(\uparrow s_1)(\uparrow s_2)) & = & m(\{t \mid s_1 \leq s_2 \to t\}) & \text{by 16.2.13(i)} \\ & = & \{t' \mid \exists t. s_1 \leq s_2\to t \;\&\; t \,\mu\, t'\} & \\ & = & \{t' \mid \exists r'. s_1 \,\mu\, (r'\to t') \;\&\; s_2 \,\mu\, r'\} & \text{by (†)} \\ & = & \{t' \mid \exists r'_1 \in \{r' \mid s_2 \,\mu\, r'\}. r'_1 \to t' \in \{t'_1 \mid s_1 \,\mu\, t'_1\}\} & \\ & = & F^{\mathcal{S}'}(\{t' \mid s_1 \,\mu\, t'\})(\{r' \mid s_2 \,\mu\, r'\}) & \\ & = & F^{\mathcal{S}'}(m(\uparrow s_1))(m(\uparrow s_2)) & \end{array}$$

From the previous two propositions we get easily

16.2.16. Theorem. Flt *induces a functor from* **NTS** *to* **NLS**.

In the following we show the dual result: functor Comp restricts to a functor from **NLS** to **NTS**.

16.2.17. Definition. Let $\mathcal{D}$ be a λ-structure. Then $\mathsf{Comp}(\mathcal{D}) = (\mathcal{K}(\mathcal{D}), \leq)$ $\in$**MSL** can be turned into an intersection type structure with top $\Omega = \perp$, by defining for $d, e \in \mathcal{K}(\mathcal{D})$

$$d \leadsto e = G(d \Rightarrow e).$$

Lemma 16.2.11 states that $d \leadsto e \in \mathcal{K}(\mathcal{D})$.

16.2.18. Proposition. *Let* $\mathcal{D} \in \mathbf{NLS}$. *Then* $\mathcal{K}(\mathcal{D}) \in \mathbf{NTS}$.

PROOF. As to (Ω-η) we have:

$$
\begin{array}{rcll}
\bot \Rightarrow \bot \sqsubseteq F(\bot) & \Rightarrow & G(\bot \Rightarrow \bot) \sqsubseteq G(F(\bot)) & \\
& \Rightarrow & G(\bot \Rightarrow \bot) \sqsubseteq \bot, & \text{since } G \circ F \sqsubseteq \mathbf{1}_{\mathcal{D}}, \\
& \Rightarrow & G(\bot \Rightarrow \bot) = \bot, & \text{following directly by 16.2.11(iii),} \\
& \Rightarrow & \bot \leadsto \bot = \bot &
\end{array}
$$

As to (η), for $a, b, a', b' \in \mathcal{K}(\mathcal{D})$ one has

$$
\begin{array}{rcl}
a' \leq a \;\&\; b \leq b' & \Rightarrow & a \sqsubseteq a' \;\&\; b' \sqsubseteq b \\
& \Rightarrow & a' \Rightarrow b' \sqsubseteq a \Rightarrow b \\
& \Rightarrow & G(a' \Rightarrow b') \sqsubseteq G(a \Rightarrow b) \\
& \Rightarrow & a' \leadsto b' \sqsubseteq a \leadsto b \\
& \Rightarrow & a \leadsto b \leq a' \leadsto b'.
\end{array}
$$

As to ($\rightarrow$-$\cap$), let $a, b, c \in \mathcal{K}(\mathcal{D})$. Then for all $d \in \mathcal{D}$ one has

$$((a \Rightarrow b) \sqcup (a \Rightarrow c))(d) = (a \Rightarrow (b \sqcup c))(d),$$

since both sides equal $b \sqcup c$ if $a \sqsubseteq d$ and $\bot$ otherwise. Therefore

$$
\begin{array}{rrcll}
& \multicolumn{3}{l}{(a \Rightarrow b) \sqcup (a \Rightarrow c) = a \Rightarrow (b \sqcup c),} & \\
\Rightarrow & G(a \Rightarrow b) \sqcup G(a \Rightarrow c) & = & G(a \Rightarrow (b \sqcup c)), & \text{by lemma 16.2.11(ii),} \\
\Rightarrow & (a \leadsto b) \sqcup (a \leadsto c) & = & a \leadsto (b \sqcup c) & \text{in } \mathcal{D}, \\
\Rightarrow & (a \leadsto b) \cap (a \leadsto c) & = & a \leadsto (b \cap c) & \text{in } \mathcal{K}(\mathcal{D}). \blacksquare
\end{array}
$$

16.2.19. PROPOSITION. *Let* $m \in \mathbf{NLS}(\mathcal{D}, \mathcal{D}')$. *Then* $\mathsf{Cmp}(m) \in \mathbf{NTS}(\mathcal{K}(\mathcal{D}), \mathcal{K}(\mathcal{D}'))$.

PROOF. We have to show that $\mu = \mathsf{Cmp}(m)$ satisfies condition (†) of Definition 16.2.1. Remember that for $d \in \mathcal{K}(\mathcal{D}), d' \in \mathcal{K}(\mathcal{D}')$

$$d \, \mu \, d' \;\Leftrightarrow\; d' \sqsubseteq m(d).$$

So we have to prove that for all $d_1, d_2 \in \mathcal{K}(\mathcal{D})$, $d' \in \mathcal{K}(\mathcal{D}')$,

$\exists e' \in \mathcal{K}(\mathcal{D}').e' \sqsubseteq m(d_2) \;\&\; G'(e' \Rightarrow d') \sqsubseteq m(d_1) \;\Leftrightarrow$
$\exists e \in \mathcal{K}(\mathcal{D}).d' \sqsubseteq m(e) \;\&\; G(d_2 \Rightarrow e) \sqsubseteq d_1.$

As to ($\Rightarrow$), we have

$$
\begin{array}{rcll}
m(F(d_1)(d_2)) & = & F'(m(d_1))(m(d_2)), & \text{since } m \in \mathbf{NLS}(\mathcal{D}, \mathcal{D}'), \\
& \sqsupseteq & F'(G'(e' \Rightarrow d'))(e'), & \text{by hypothesis,} \\
& \sqsupseteq & (e' \Rightarrow d')(e'), & \text{since } F' \circ G' \sqsupseteq \mathbf{1}_{[\mathcal{D}' \to \mathcal{D}']}, \\
& = & d'. &
\end{array}
$$

Since m is continuous and e' is compact, there exists $e \in \mathcal{K}(\mathcal{D})$ such that

$$e \sqsubseteq F(d_1)(d_2) \;\&\; d' \sqsubseteq m(e).$$

Therefore we have

$$\begin{array}{rcll} e \sqsubseteq F(d_1)(d_2) & \Rightarrow & d_2 \Rightarrow e \sqsubseteq F(d_1) & \\ & \Rightarrow & G(d_2 \Rightarrow e) \sqsubseteq d_1, & \text{since } G \circ F \sqsubseteq \mathbf{1}_{\mathcal{D}}. \end{array}$$

As to ($\Leftarrow$), we have

$$\begin{array}{rcll} G(d_2 \Rightarrow e) \sqsubseteq d_1 & \Rightarrow & d_2 \Rightarrow e \sqsubseteq F(d_1), & \text{since } G \circ F \sqsupseteq \mathbf{1}_{\mathcal{D}}, \\ & \Rightarrow & e \sqsubseteq F(d_1)(d_2) & \\ & \Rightarrow & m(e) \sqsubseteq m(F(d_1)(d_2)) & \\ & \Rightarrow & m(e) \sqsubseteq F'(m(d_1))(m(d_2)) & \text{since } m{\in}\mathbf{NLS}(\mathcal{D},\mathcal{D}') \\ & \Rightarrow & d' \sqsubseteq F'(m(d_1))(m(d_2)), & \text{since } d' \sqsubseteq m(e). \end{array}$$

Since $F'(m(d_1))$ is a continuous function and d' is compact, there exists $e'{\in}\mathcal{K}(\mathcal{D}')$ such that

$$e' \sqsubseteq m(d_2) \;\&\; d' \sqsubseteq F'(m(d_1))(e').$$

Finally we have

$$\begin{array}{rcll} d' \sqsubseteq F'(m(d_1))(e') & \Rightarrow & e' \Rightarrow d' \sqsubseteq F'(m(d_1)) & \\ & \Rightarrow & G'(e' \Rightarrow d') \sqsubseteq m(d_1) & \text{since } G' \circ F' \sqsubseteq \mathbf{1}_{\mathcal{D}'}. \blacksquare \end{array}$$

From the previous two propositions we easily get the following.

16.2.20. THEOREM. Cmp *induces a functor from* **NLS** *to* **NTS**.

From Theorems 16.2.16 and 16.2.20 it follows that if $\mathcal{S}{\in}\mathbf{NTS}$, then so is $\mathcal{K}(\mathcal{F}^{\mathcal{S}})$, and that if $\mathcal{D}{\in}\mathbf{NLS}$, then so is $\mathcal{F}^{\mathcal{K}(D)}$. We will show in the next proposition that these pairs of structures are even isomorphic.

16.2.21. PROPOSITION. (i) *Let* $\mathcal{D}{\in}\mathbf{NLS}$. *Define* $m : \mathcal{F}^{\mathcal{K}(\mathcal{D})}{\to}\mathcal{D}$ *by*

$$m(X) = \bigsqcup X.$$

Then m is an NLS isomorphism.

(ii) *Let* $\mathcal{S}{\in}\mathbf{NTS}$. *Define* $\varphi : \mathcal{S}{\to}\mathcal{K}(\mathcal{F}^{\mathcal{S}})$ *by*

$$\varphi(s) = {\uparrow}s.$$

Then φ is an NTS isomorphism.

PROOF. (i) By proposition 16.1.17(ii) is a lattice isomorphism. By the continuity of m it suffices to show for $c, d{\in}\mathcal{K}(\mathcal{D})$

$$m({\uparrow}c.{\uparrow}d) = m({\uparrow}c) \cdot m({\uparrow}d).$$

Remark that

$$\begin{array}{rcll} c \leq_{\mathcal{K}(\mathcal{D})} d \leadsto b & \Leftrightarrow & G(d \Rightarrow b) \sqsubseteq c, & \text{by definition of } \leq_{\mathcal{K}(\mathcal{D})} \text{ and } \leadsto, \\ & \Leftrightarrow & d \Rightarrow b \sqsubseteq F(c), & \text{since } F \circ G \sqsupseteq \mathbf{1}_{[\mathcal{D}\to\mathcal{D}]}, \\ & \Leftrightarrow & b \sqsubseteq F(c)(d), & \text{by definition of } \Rightarrow, \\ & \Leftrightarrow & b \sqsubseteq c \cdot d. & \end{array}$$

Now we have

$$\begin{array}{rcll} m(\uparrow c.\uparrow d) & = & m(\uparrow\{b \mid \exists a{\in}\uparrow d.a \rightsquigarrow b{\in}\uparrow c\}) & \\ & = & m(\uparrow\{b \mid \exists a \geq d.a \rightsquigarrow b \geq c\}) & \\ & = & m(\uparrow\{b \mid d \rightsquigarrow b \geq c\}), & \text{by lemma 16.2.7(ii),} \\ & = & m(\{b \mid d \rightsquigarrow b \geq c\}), & \text{by that same lemma,} \\ & = & \bigsqcup\{b \mid b \sqsubseteq c \cdot d\}, & \text{by the remark above,} \\ & = & c \cdot d & \\ & = & m(\uparrow c) \cdot m(\uparrow d), & \end{array}$$

since $m(\uparrow c) = \bigsqcup\{e \mid c \leq e\} = \bigsqcup\{e \mid e \sqsubseteq c\} = c$.

(ii) Since the map $\varphi : \mathcal{S} \to \mathcal{K}(\mathcal{F}^{\mathcal{S}})$ is an isomorphism in the category $\mathbf{MSL}_0$ by proposition 16.1.17(i), we can use proposition 16.2.3 in order to show that φ induces an isomorphism in **NTS**. So it is enough to prove that condition $(*)$ of that proposition holds, that is, for any $s_1, s_2, t{\in}S$:

$$s_1 \leq s_2 \to t \;\Leftrightarrow\; \varphi(s_1) \leq \varphi(s_2) \rightsquigarrow \varphi(t).$$

We have

$$\begin{array}{rcll} \varphi(s_1) \leq \varphi(s_2) \rightsquigarrow \varphi(t) & \Leftrightarrow & \uparrow s_1 \leq \uparrow s_2 \rightsquigarrow \uparrow t & \\ & \Leftrightarrow & \uparrow s_2 \rightsquigarrow \uparrow t \subseteq \uparrow s_1 & \\ & \Leftrightarrow & G^{\mathcal{S}}(\uparrow s_2 \Rightarrow \uparrow t) \subseteq \uparrow s_1, & \text{by definition of } \rightsquigarrow, \\ & \Leftrightarrow & \uparrow(s_2 \to t) \subseteq \uparrow s_1, & \text{by Lemma 16.2.13(ii),} \\ & \Leftrightarrow & s_1 \leq s_2 \to t. & \blacksquare \end{array}$$

16.2.22. THEOREM. *The categories* **NTS** *and* **NLS** *are equivalent. In fact the isomorphisms in 16.2.21 form natural isomorphisms showing*

$$\mathsf{Cmp} \circ \mathsf{Flt} \cong \mathbf{1}_{\mathbf{NTS}} \;\&\; \mathsf{Flt} \circ \mathsf{Cmp} \cong \mathbf{1}_{\mathbf{NLS}}.$$

PROOF. After 16.2.21 it remains to prove naturality. This follows exactly along the lines of the proof of corollary 16.1.19. ■

Representability of continuous functions

In this subsection we will isolate a number of conditions on an $\mathcal{S}{\in}\mathbf{NTS}$ implying that the set of *representable functions* $\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}$, i.e. the set of functions in the image of $F^{\mathcal{S}}$, contains several interesting classes of functions.

16.2.23. LEMMA. *Let* $S{\in}\mathbf{NTS}$ *and let* $f{\in}[\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}]$. *Then*

$$f \text{ is representable } \;\Leftrightarrow\; F^{\mathcal{S}} \circ G^{\mathcal{S}}(f) = f.$$

PROOF. $(\Rightarrow)$ Trivial. $(\Leftarrow)$ Suppose $F^{\mathcal{S}}(X) = f$. Claim $F^{\mathcal{S}}(G^{\mathcal{S}}(F^{\mathcal{S}}(X))) = F^{\mathcal{S}}(X)$. As in the proof of proposition 16.2.14 one has $G^{\mathcal{S}}(F^{\mathcal{S}}(X)) = \uparrow\{a \to b \mid a \to b{\in}X\}$. Hence $a \to b{\in}G^{\mathcal{S}}(F^{\mathcal{S}}(X)) \;\Leftrightarrow\; a \to b{\in}X$. So for each Y we have indeed $F^{\mathcal{S}}(G^{\mathcal{S}}(F^{\mathcal{S}}(X)))(Y) = F^{\mathcal{S}}(X)(Y)$. ■

16.2.24. LEMMA. *Let* $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{S}$ *and let*

$$h = (\uparrow a_1 \Rightarrow \uparrow b_1) \sqcup \dots \sqcup (\uparrow a_n \Rightarrow \uparrow b_n).$$

Then for $c \in \mathcal{S}$

(i) $h(\uparrow c) = \{d \mid \bigcap\{b_i \mid c \leq a_i\} \leq d\}$.

(ii) $(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)(\uparrow c) = \{d \mid (a_1 \to b_1) \cap \dots \cap (a_n \to b_n) \leq (c \to d)\}$.

16.2.25. THEOREM. *let* $* \in \mathbf{NTS}$. *Let* $\mathcal{R}$ *be the set of representable functions* $\mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}$. *Then we have the following.*

(i) $\mathcal{R}$ *contains the bottom function* $\mathsf{K}\bot$ *iff for all* $c, d \in \mathcal{S}$

$$\Omega \leq c \to d \;\Rightarrow\; \Omega \leq d.$$

(ii) $\mathcal{R}$ *contains the constant functions iff for all* $b, c, d \in \mathcal{S}$

$$\Omega \to b \leq c \to d \;\Rightarrow\; b \leq d.$$

(iii) $\mathcal{R}$ *contains the continous step functions iff for all* $a, b, c, d \in \mathcal{S}$

$$a \to b \leq c \to d \;\&\; d \neq \Omega \;\Rightarrow\; c \leq a \;\&\; b \leq d.$$

(iv) $\mathcal{R}$ *contains the continous functions iff*

$$\begin{array}{lll} \forall n \geq 0, \forall a_1, \dots, a_n, b_1, \dots, b_n, c, d \in \mathcal{S} & & \\ (a_1 \to b_1) \cap \dots (a_n \to b_n) \leq (c \to d) & \Rightarrow & [c \leq a_{i_1} \cap \dots \cap a_{i_k} \;\&\; b_{i_1} \cap \dots \cap b_{i_k} \leq d] \\ & & \textit{for some } k \geq 0,\ 1 \leq i_1 < \dots < i_k \leq n. \end{array}$$

PROOF. (i) Assume that $\mathsf{K}\bot \in \mathcal{R}$. Then $F^{\mathcal{S}}(G^{\mathcal{S}}(\mathsf{K}\bot)) = \mathsf{K}\bot$, by lemma 16.2.23. Observe that

$$\begin{array}{rcll} G^{\mathcal{S}}(\mathsf{K}\bot) & = & G^{\mathcal{S}}(\bot \Rightarrow \bot) = G^{\mathcal{S}}(\uparrow\Omega \Rightarrow \uparrow\Omega) & \\ & = & \uparrow(\Omega \to \Omega), & \text{by 16.2.13(ii)} \\ & = & \bot = \uparrow\Omega. & \end{array}$$

Hence $F^{\mathcal{S}}(\bot) = \mathsf{K}\bot$, so in particular $F^{\mathcal{S}}(\bot)(\uparrow c) = \bot$, and therefore

$$\begin{array}{rcll} \{d \mid \Omega \leq d\} & = & \bot & \\ & = & F^{\mathcal{S}}(\bot)(\uparrow c) & \\ & = & F^{\mathcal{S}}(\uparrow\Omega)(\uparrow c) & \\ & = & \{d \mid \Omega \leq (c \to d)\}, & \text{by 16.2.13(i).} \end{array}$$

But then $\Omega \leq c \to d \;\Rightarrow\; \Omega \leq d$.

($\Leftarrow$) Suppose that $\Omega \leq c \to d \;\Rightarrow\; \Omega \leq d$. We show that $F^{\mathcal{S}}(\bot) = \mathsf{K}\bot$.

$$\begin{array}{rcll} F^{\mathcal{S}}(\bot)(X) & = & \{b \mid \exists a \in X. (a \to b) \in \bot\} & \\ & = & \{b \mid \exists a \in X. (a \to b) \in \uparrow\Omega\} & \\ & = & \{b \mid \exists a \in X. \Omega \leq (a \to b)\} & \\ & = & \{b \mid \Omega \leq b\}, & \text{by the assumption,} \\ & = & \uparrow\Omega = \bot. & \end{array}$$

(ii) Suppose that $\Omega{\to}b \leq c{\to}d \;\Rightarrow\; b \leq d$. We first show that each compact constant function $\mathsf{K}\!\uparrow\! b (= \lambda c{\in}\mathcal{F}^{\mathcal{S}}.{\uparrow} b)$ is represented by $\uparrow(\Omega{\to}b)$. Indeed,

$$\begin{array}{lcll} d{\in}{\uparrow}(\Omega{\to}b)\cdot{\uparrow}c & \Leftrightarrow & c{\to}d{\in}{\uparrow}(\Omega{\to}b), & \text{by } (\eta), \\ & \Leftrightarrow & \Omega{\to}b \leq c{\to}d & \\ & \Leftrightarrow & b \leq d, & \text{by the assumption, } (\Omega) \text{ and } (\eta), \\ & \Leftrightarrow & d{\in}{\uparrow}b = (\mathsf{K}\!\uparrow\! b)\,({\uparrow}c). & \end{array}$$

Now let $\mathsf{K}\,X$ be an arbitrary constant function. Then $\mathsf{K}\,X = \sqcup_{b\in X}\mathsf{K}\!\uparrow\! b$ and $\{\mathsf{K}\!\uparrow\! b \mid b{\in}X\}$ is directed. Hence by the previous lemma and the continuity of $F^{\mathcal{S}} \circ G^{\mathcal{S}}$ we get

$$\begin{array}{lcl} \mathsf{K}\,X & = & \sqcup_{b\in X}\mathsf{K}\!\uparrow\! b \\ & = & \sqcup_{b\in X} F^{\mathcal{S}} \circ G^{\mathcal{S}}(\mathsf{K}\!\uparrow\! b) \\ & = & F^{\mathcal{S}} \circ G^{\mathcal{S}}(\sqcup_{b\in X}\mathsf{K}\!\uparrow\! b). \end{array}$$

Conversely, suppose that all constant functions are representable. Then $F^{\mathcal{S}} \circ G^{\mathcal{S}}(\mathsf{K}\!\uparrow\! b) = \mathsf{K}\!\uparrow\! b$, by the previous lemma. Therefore

$$\begin{array}{lcll} \Omega{\to}b \leq c{\to}d & \Rightarrow & (c{\to}d){\in}{\uparrow}(\Omega{\to}b) & \\ & \Rightarrow & d{\in}{\uparrow}(\Omega{\to}b)\cdot{\uparrow}c & \\ & \Rightarrow & d{\in}((F^{\mathcal{S}} \circ G^{\mathcal{S}})(\mathsf{K}\!\uparrow\! b))({\uparrow}c), & \text{by 16.2.13(ii)}, \\ & \Rightarrow & d{\in}(\mathsf{K}\!\uparrow\! b)({\uparrow}c) = {\uparrow}b & \\ & \Rightarrow & b \leq d. & \end{array}$$

(iv) (We prove (iii) later.) Let $\mathcal{S}{\in}\mathbf{NTS}$. Using the axioms of the natural type structure $\mathcal{S}$, it is not difficult to see that the condition in the RHS of (iv) is equivalent with

$$(a_1{\to}b_1) \cap \ldots \cap (a_n{\to}b_n) \leq (c{\to}d) \;\Leftrightarrow\; \bigcap\{b_i \mid c \leq a_i\} \leq d. \qquad (*)$$

($\Leftarrow$ always does hold) Let $h{\in}\mathcal{K}([\mathcal{F}^{\mathcal{S}}{\to}\mathcal{F}^{\mathcal{S}}])$. By proposition 16.2.9(iii) one has that h is the sup of some $A \Rightarrow B$ with $A, B{\in}\mathcal{K}(\mathcal{F}^{\mathcal{S}})$ and hence by the compactness of h the sup of finitely many such step-functions. Therefore for some $a_1, \ldots, a_n, b_1, \ldots, b_n{\in}\mathcal{S}$

$$h = ({\uparrow}a_1 \Rightarrow {\uparrow}b_1) \sqcup \ldots \sqcup ({\uparrow}a_n \Rightarrow {\uparrow}b_n),$$

since a compact $A{\in}\mathcal{F}^{\mathcal{S}}$ is of the form $A = {\uparrow}a$ with $a{\in}\mathcal{S}$.

We want to compute $h({\uparrow}c)$ and $(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)({\uparrow}c)$.

$$\begin{array}{lcll} h({\uparrow}c) & = & \bigsqcup\{{\uparrow}b_i \mid {\uparrow}a_i \leq {\uparrow}c\} & \\ & = & {\uparrow}\bigcap\{b_i \mid c \leq a_i\}, & \text{by 16.1.13(iii)} \\ & = & \{d \mid \bigcap\{b_i \mid c \leq a_i\} \leq d\}; & \end{array}$$

$$
\begin{array}{rll}
(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)(\uparrow c) & = & \\
& = F^{\mathcal{S}}(G^{\mathcal{S}}(\uparrow(a_1 \Rightarrow b_1)) \sqcup \ldots \sqcup G^{\mathcal{S}}(\uparrow(a_n \Rightarrow b_n))))(\uparrow c), & \text{by 16.2.11(ii)},\\
& = F^{\mathcal{S}}(\uparrow(a_1 \to b_1) \sqcup \ldots \sqcup \uparrow(a_n \to b_n)))(\uparrow c), & \text{by 16.2.13(ii)},\\
& = F^{\mathcal{S}}(\uparrow((a_1 \to b_1) \cap \ldots \cap (a_n \to b_n)))(\uparrow c), & \text{by 16.1.13(iii)},\\
& = \{d \mid \exists e \in \uparrow c.(e \to d) \in \uparrow((a_1 \to b_1) \cap \ldots \cap (a_n \to b_n))\}, & \text{see 16.2.12},\\
& = \{d \mid \exists e \geq c.(a_1 \to b_1) \cap \ldots \cap (a_n \to b_n) \leq (e \to d)\} & \\
& = \{d \mid (a_1 \to b_1) \cap \ldots \cap (a_n \to b_n) \leq (c \to d)\}, & \text{by } (\eta).
\end{array}
$$

($\Rightarrow$) Suppose all continuous functions are representable. Then since h above is such a function, one has

$$(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)(\uparrow c) = h(\uparrow c)$$

and therefore (*).

($\Leftarrow$) Suppose we have (*). Then for the compact continuous functions h one has that $(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)$ and h coincide on the compact elements $\uparrow c$ and therefore by proposition 16.1.9 everywhere. But then it follows again that $f = (F^{\mathcal{S}} \circ G^{\mathcal{S}})(f)$ for every continuous $f : \mathcal{F}^{\mathcal{S}} \to \mathcal{F}^{\mathcal{S}}$. Hence lemma 16.2.23 applies.

(ii) ($\Rightarrow$) Suppose all continuous stepfunctions are representable. Suppose $a \to b \leq c \to d$, $d \neq \top$. Take $h = \uparrow a \Rightarrow \uparrow b$. As in the proof of (iii) we compute

$$
\begin{array}{rcl}
(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)(\uparrow c) & = & \{e \mid a \to b \leq c \to e\}\\
h(\uparrow c) & = & \{e \mid \bigcap\{b \mid c \leq a\} \leq e\}.
\end{array}
$$

By the first assumption these two sets are equal. By the second assumption it follows that $c \leq a$ & $b \leq d$.

($\Leftarrow$) Let $h = X \Rightarrow Y$ be continuous. We have to show that

$$(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h) = h \qquad (*)$$

By lemma 16.1.9(ii) it suffices to show this for compact h. If $Y \neq \bot$, then by 16.2.7 both X, Y are compact, so $h = \uparrow a \Rightarrow \uparrow b$. Then (*) holds by lemma 16.2.24 and the assumption. If $Y = \bot$, then h is the bottom function and hence representable (the assumption in (iii) implies the assumption in (n)). ■

16.3. The strict story 26.2.2004:1018

16.3.1. DEFINITION. The category $\mathbf{MSL}^s$ of *strict* meet semi-lattices is defined as follows.

(i) The set of objects of $\mathbf{MSL}^s$ consists of countable meet semi-lattices (without the requirement that there be a top).

(ii) The set of morphisms $\mathbf{MSL}^s(\mathcal{S}, \mathcal{S}')$ consists of relations $\mu \subseteq \mathcal{S} \times \mathcal{S}'$ such that for all $s, t \in \mathcal{S}$ and $s', t', t_1', t_2' \in \mathcal{S}'$ one has

(1) $t \leq s \;\mu\; s' \leq t' \;\Rightarrow\; t \;\mu\; t'$;

(2) $s \;\mu\; t_1' \;\&\; s \;\mu\; t_2' \;\Rightarrow\; s \;\mu\; (t_1' \cap t_2')$.

Again such μ is called an approximable map and we write $\mathcal{S} - \mu - \mathcal{S}'$.

16.3.2. DEFINITION. (i) $\mathbf{MSL}^s_0$ is the category with the same objects as $\mathbf{MSL}^s$ but as morphisms maps preserving $\leq$ and $\cap$.

(ii) For $f \in \mathbf{MSL}^s_0(\mathcal{S},\mathcal{S}')$ we define $\mu_f \in \mathbf{MSL}^s(\mathcal{S},\mathcal{S}')$ by

$$s\,\mu_f\,s' \;\Leftrightarrow\; f(s) \leq s'.$$

16.3.3. LEMMA. *Let $\mu \in \mathbf{MSL}^s(\mathcal{S},\mathcal{S}')$ be an isomorphism with inverse ν. Then there exist $f, g \in \mathbf{MSL}^s_0(\mathcal{S},\mathcal{S}')$ such that f is an $\mathbf{MSL}^s_0$ isomorphism with inverse g and $\mu = \mu_f$, $\nu = \mu_g$.*

16.3.4. COROLLARY. *Let $\mathcal{S}, \mathcal{S}'$ be two objects in $\mathbf{MSL}^s$. Then*

$$\mathcal{S} \cong \mathcal{S}' \textit{ in } \mathbf{MSL}^s_0 \;\Leftrightarrow\; \mathcal{S} \cong \mathcal{S}' \textit{ in } \mathbf{MSL}^s.$$

16.3.5. DEFINITION. The category $\mathbf{ALG}^s$ is defined as follows.

(i) The objects of $\mathbf{ALG}^s$ are the same as the objects of $\mathbf{ALG}$ (ω-algebraic complete lattices).

(ii) A map $f : \mathcal{D} \to \mathcal{D}'$ is called *strict* iff $f(\bot_{\mathcal{D}}) = \bot_{\mathcal{D}'}$.

(iii) $[D \to^s \mathcal{D}'] = \{f \in [\mathcal{D} \to \mathcal{D}'] \mid f$ is continuous and strict$\}$.

(iv) The morphisms of $\mathbf{ALG}^s$ are defined as the strict Scott-continuous maps

$$\mathbf{ALG}^s(\mathcal{D},\mathcal{D}') = [\mathcal{D} \to^s \mathcal{D}'].$$

16.3.6. DEFINITION. (i) Given an $\mathcal{S} \in \mathbf{MSL}^s$, a *strict filter* over $\mathcal{S}$ is a (possibly empty) set $X \subseteq S$ such that the following hold.

(1) $s \in X \;\&\; s \leq s' \;\Rightarrow\; s' \in X$;

(2) $s, s' \in X \;\Rightarrow\; s \cap s' \in X$.

(ii) Write $\mathcal{F}^{\mathcal{S}}_s = \{X \subseteq S \mid X$ is a strict filter over $S\}$.

16.3.7. DEFINITION. Let $\mathcal{S} \in \mathbf{MSL}^s$. For $X \subseteq S$ define $\uparrow^s X$ to be the least strict filter over S containing X. The filter $\uparrow^s X$ can be described explicitly by

$$\uparrow^s X = \{x \in S \mid \exists x_1, \ldots, x_n \in X . x \geq x_1 \sqcup \ldots \sqcup x_n\}.$$

The difference between $\uparrow^s X$ and $\uparrow X$ is only for the empty set:

$$\begin{array}{rcl} \uparrow\emptyset & = & \{\top\} \\ \uparrow^s\emptyset & = & \emptyset \end{array}$$

16.3.8. PROPOSITION. *Let $S \in \mathbf{MSL}^s$.*

(i) $\mathcal{F}^{\mathcal{S}}_s = \langle \mathcal{F}^{\mathcal{S}}_s, \subseteq \rangle$ is a complete lattice, with for $\mathcal{X} \subseteq \mathcal{F}^{\mathcal{S}}_s$ the sup is

$$\bigsqcup \mathcal{X} = \uparrow^s(\cup \mathcal{X}).$$

(ii) For $t \in S$ one has $\uparrow^s t = \uparrow^s\{t\}$ and $\uparrow^s t \in \mathcal{F}^{\mathcal{S}}_s$.

(iii) For $t, t' \in S$ one has $\uparrow^s t \sqcup \uparrow^s t' = \uparrow^s(t \cap t')$.

(iv) *For* $X \in \mathcal{F}_s^{\mathcal{S}}$ *one has*

$$\begin{array}{rclcl} X &=& \bigsqcup\{\uparrow^s t \mid t \in X\} &=& \bigsqcup\{\uparrow^s t \mid \uparrow^s t \subseteq X\} \\ &=& \bigcup\{\uparrow^s t \mid t \in X\} &=& \bigcup\{\uparrow^s t \mid \uparrow^s t \subseteq X\}. \end{array}$$

(v) $\{\uparrow^s t \mid t \in S\} \cup \{\emptyset\}$ *is the set of finite elements of* $\mathcal{F}_s^{\mathcal{S}}$.
(vi) $\mathcal{F}_s^{\mathcal{S}} \in \mathbf{ALG}^s$.

16.3.9. DEFINITION. The functor $\mathsf{Flt} : \mathbf{MSL}^s \to \mathbf{ALG}^s$ is defined as follows.

$$\mathsf{Flt}(S) = \langle \mathcal{F}_s^{\mathcal{S}}, \subseteq \rangle.$$

For an arrow $\mu \in \mathbf{MSL}^s(\mathcal{S}, \mathcal{S}')$ we define $\mathsf{Flt}(\mu) \in \mathbf{ALG}^s(\mathcal{F}_s^{\mathcal{S}}, \mathcal{F}_s^{S'})$ by

$$\mathsf{Flt}(\mu)(X) = \mu(X)$$

16.3.10. DEFINITION. (i) Let $\mathcal{D} \in \mathbf{ALG}^s$. Write $\mathcal{K}^s(\mathcal{D}) = \mathcal{K}(\mathcal{D})/\perp_{\mathcal{D}}$.
(ii) The functor $\mathsf{Cmp} : \mathbf{ALG}^s \to \mathbf{MSL}^s$ is defined as follows

$$\mathsf{Cmp}((D, \sqsubseteq)) = (\mathcal{K}^s(\mathcal{D}), \leq),$$

with for $a, b \in \mathcal{K}^s(D)$

$$a \leq b \;\Leftrightarrow\; b \sqsubseteq a.$$

(iii) For $f : D \to D'$ define the approximable map $\mathcal{K}^s(D) - \mathtt{Cmp}(f) - \mathcal{K}^s(D')$

$$c \;\mathsf{Cmp}(f)\; c' \;\Leftrightarrow\; c' \sqsubseteq f(c).$$

16.3.11. LEMMA. $\mathcal{K}(D) \in \mathbf{MSL}^s$ *with*

$$\begin{array}{rcl} d \cap_{\leq} e &=& d \sqcup_{\sqsubseteq} e; \\ \perp_{\leq} &=& \top_{\sqsubseteq}. \end{array}$$

16.3.12. PROPOSITION. (i) *Let* $\mathcal{S} \in \mathbf{MSL}^s$. *Then* $\varphi : \mathcal{S} \to \mathcal{K}^s(\mathcal{F}_s^{\mathcal{S}})$ *defined by* $\varphi(s) = \uparrow s$ *is an* $\mathbf{MSL}_0$ *isomorphism. Hence also in* $\mathbf{MSL}^s$ *one has* $\mathcal{S} \cong \mathcal{K}^s(\mathcal{F}_s^{\mathcal{S}})$.

$$\begin{tikzcd} \mathcal{S} \arrow[rr, dotted, "\mathsf{Flt}"] \arrow[dr, "\uparrow"'] & & \mathcal{F}_s^{\mathcal{S}} \arrow[dl, dotted, "\mathsf{Cmp}"] \\ & \mathcal{K}^s(\mathcal{F}_s^{\mathcal{S}}) & \end{tikzcd}$$

(ii) *Let* $\mathcal{D} \in \mathbf{ALG}^s$. *Then* $m : \mathcal{F}_s^{\mathcal{K}^s(\mathcal{D})} \to \mathcal{D}$ *defined by* $m(X) = \bigsqcup X$, *is an* $\mathbf{ALG}^s$ *isomorphism with inverse* $n : \mathcal{D} \to \mathcal{F}_s^{\mathcal{K}^s(\mathcal{D})}$ *defined by* $n(d) = \{c \in \mathcal{K}^s(\mathcal{D}) \mid c \sqsubseteq d\}$.

$$\begin{tikzcd} \mathcal{K}^s(\mathcal{D}) \arrow[dr, dotted, "\mathsf{Flt}"'] & & \mathcal{D} \arrow[ll, dotted, "\mathsf{Cmp}"'] \\ & \mathcal{F}_s^{\mathcal{K}^s(\mathcal{D})} \arrow[ur, "\bigsqcup"'] & \end{tikzcd}$$

16.3.13. COROLLARY. *The categories* $\mathbf{MSL}^s$ *and* $\mathbf{ALG}^s$ *are equivalent.*

Natural Strict Type Structures and Lambda Structures

This subsection is similar to section 16.2

A strict intersection type structure is a strict meet semi-lattice with an additional operation $\to$ on its domain

$$\mathcal{S} = \langle S, \leq, \cap, \to \rangle.$$

16.3.14. Definition. (i) A *restricted natural* type structure $\mathcal{M}$ is a strict intersection type structure such that $\mathcal{M}$ satisfies (η) and $(\to\text{-}\cap)$.

$(\to\text{-}\cap)$	$(A\to B)\cap(A\to C) \leq A\to B\cap C$
(η)	$A' \leq A \ \&\ B \leq B' \ \Rightarrow\ A\to B \leq A'\to B'$

(ii) The category $\mathbf{NTS}^s$ has as objects the restricted natural type structures.

(iii) The morphisms of $\mathbf{NTS}^s$ are approximable mappings $\mu\in\mathbf{MSL}^s(\mathcal{S},\mathcal{S}')$ which satisfy the following condition. For all $s_1, s_2\in S$, $t'\in S'$,

$$(\dagger)\ \ \exists s'\in S'.[s_2\,\mu\,s' \text{ and } s_1\,\mu\,(s'\to t')] \ \Leftrightarrow\ \exists t\in S.[s_1 \leq s_2\to t \text{ and } t\,\mu\,t'].$$

Notice that the condition above specializes to $\mathbf{MSL}_0^s$ morphisms as follows: if $\mu = \mu_f$, with $f : S\to S'$, then $\mu_f\in\mathbf{NTS}^s(\mathcal{S},\mathcal{S}')$ iff for all $s_1, s_2\in S$, $t'\in S'$,

$$(\ddagger)\ \ f(s_1) \leq f(s_2)\to t' \ \Leftrightarrow\ \exists t\in S.s_1 \leq s_2\to t \text{ and } f(t) \leq t'.$$

16.3.15. Proposition. *Let $\mu\in\mathbf{MSL}^s(\mathcal{S},\mathcal{S}')$. Then μ is an $\mathbf{NTS}^s$-isomorphism if and only if $\mu = \mu_f$, with $f : S\to S'$ monotone bijection with inverse f^{-1} monotone, and such that, for each $s_1, s_2, t\in S$, we have:*

$$(*)\ \ s_1 \leq s_2\to t \ \Leftrightarrow\ f(s_1) \leq f(s_2)\to f(t).$$

16.3.16. Proposition. *$\mathcal{S}$ is a restricted natural type structure if and only if for each $r\in S$, the relation $\mu_r \subseteq S\times S$ defined by*

$$s_1\,\mu_r\,t_1 \ \Leftrightarrow\ r \leq s_1\to t_1,$$

is an approximable mapping.

16.3.17. Definition. (i) If $\mathcal{D}\in\mathbf{ALG}^s$, then $[\mathcal{D}\to^s\mathcal{D}']$ denotes the set of continuous functions from $\mathcal{D}$ to $\mathcal{D}'$.

(ii) $[\mathcal{D}\to^s\mathcal{D}]$ is partially ordered pointwise:

$$f \sqsubseteq g \ \Leftrightarrow\ \forall x\in\mathcal{D}.f(x) \sqsubseteq g(x).$$

(iii) If $a, b\in\mathcal{D}$, then $a \Rightarrow b$ is the *strict step function* defined as before but with the extra condition that $a = \bot$ implies $b = \bot$.

16.3.18. Lemma. *$[\mathcal{D}\to^s\mathcal{D}]$ is a complete lattice with*

$$(\bigsqcup_{f\in X} f)(d) = \bigsqcup_{f\in X} f(d).$$

16.3.19. PROPOSITION. *Let* $\mathcal{D}\in\mathbf{ALG}^s$.

(i) *If* $a, b\in\mathcal{D}$ *with* a *compact and* $a = \bot$ *implies* $b = \bot$, *then* $a \Rightarrow b\in[\mathcal{D}\to^s\mathcal{D}]$.

(ii) *Let* $a, b\in\mathcal{D}$ *with* a, b *compact. Suppose* $a = \bot \Rightarrow b = \bot$. *Then* $a \Rightarrow b$ *is compact in* $[\mathcal{D}\to^s\mathcal{D}]$.

(iii) *For* $f\in[\mathcal{D}\to^s\mathcal{D}]$ *one has* $f = \bigsqcup\{a \Rightarrow b \mid b \sqsubseteq f(a),\ a, b \text{ compact in } \mathcal{D}\}$.

(iv) *The compact elements in* $[\mathcal{D}\to^s\mathcal{D}]$ *are directed.*

(v) $[\mathcal{D}\to^s\mathcal{D}]\in\mathbf{ALG}^s$.

16.3.20. DEFINITION. (i) A *strict* λ*-structure* is a triple $\langle\mathcal{D}, F, G\rangle$, where $\mathcal{D}$ is in $\mathbf{ALG}^s$ and $F : [\mathcal{D}\to^s[\mathcal{D}\to^s\mathcal{D}]]$, $G : [[\mathcal{D}\to^s\mathcal{D}]\to^s\mathcal{D}]$ are Scott continuous functions.

(ii) In a strict λ-structure one defines

$$d \cdot e \quad = \quad F(d)(e).$$

(iii) A *restricted natural* λ-structure is one that satisfies

$$\begin{aligned} F \circ G &\sqsupseteq \mathbf{1}_{\mathcal{D}\to\mathcal{D}} \\ G \circ F &\sqsubseteq \mathbf{1}_{\mathcal{D}} \end{aligned}$$

(iv) The category $\mathbf{NLS}^s$ has as objects restricted natural lambda structures and as morphisms strict Scott continuous maps m satisfying

$$m(d \cdot e) \quad = \quad m(d) \cdot m(e).$$

There is no need for a condition on m in terms of G, as we will see that F completely determines G.

16.3.21. LEMMA. *Let* $\mathcal{D}\in\mathbf{NLS}^s$.

(i) *For* $f\in[\mathcal{D}\to^s\mathcal{D}]$

$$f \text{ is compact} \ \Rightarrow\ G(f) \text{ is compact.}$$

(ii) *For* $f_1, f_2\in[\mathcal{D}\to^s\mathcal{D}]$

$$G(f_1 \sqcup f_2) = G(f_1) \sqcup G(f_2).$$

(iii) F *determines* G*: for all* $f\in[\mathcal{D}\to^s\mathcal{D}]$

$$G(f) = \sqcap\{d \mid f \sqsubseteq F(d)\}.$$

16.3.22. DEFINITION (Strict filter structure). Let $\mathcal{S}$ be an object of $\mathbf{NTS}^s$.

(i) On $\mathcal{F}^{\mathcal{S}}_s\in\mathbf{ALG}^s$ we define $F^{\mathcal{S}}\in[\mathcal{F}^{\mathcal{S}}\to^s[\mathcal{F}^{\mathcal{S}}\to^s\mathcal{F}^{\mathcal{S}}]]$ and $G^{\mathcal{S}} : [[\mathcal{F}^{\mathcal{S}}\to^s\mathcal{F}^{\mathcal{S}}]\to^s\mathcal{F}^{\mathcal{S}}]$ as follows.

$$\begin{aligned} F^{\mathcal{S}}(X)(Y) &= \{b\in\mathcal{S} \mid \exists a\in Y. a \to b\in X\} \\ G^{\mathcal{S}}(f) &= \uparrow\{a\to b\in\mathcal{S} \mid b\in f(\uparrow a)\} \end{aligned}$$

By (η) and $(\to\text{-}\cap)$ it follows that $F^{\mathcal{S}}(X)(Y)$ is actually a strict filter.

(ii) The triple $\mathcal{F}^{\mathcal{S}}_s = \langle\mathcal{F}^{\mathcal{S}}, F^{\mathcal{S}}, G^{\mathcal{S}}\rangle$ is called the *filter structure* over $\mathcal{S}$.

16.3.23. LEMMA. *Let $\mathcal{S}$ be an object of* $\mathbf{NTS}^s$ *and* $s_1, s_2 \in S$. *Then*

(i) $F^{\mathcal{S}}(\uparrow s_1)(\uparrow s_2) = \{t \in S \mid s_1 \leq s_2 \rightarrow t\}$.

(ii) $G^{\mathcal{S}}(\uparrow s_1 \Rightarrow \uparrow s_2) = \uparrow(s_1 \rightarrow s_2)$.

16.3.24. PROPOSITION. *Let $\mathcal{S}$ be in* $\mathbf{NTS}^s$. *Then* $\mathcal{F}^{\mathcal{S}}$ *is in* $\mathbf{NLS}^s$.

16.3.25. PROPOSITION. *Let* $\mu \in \mathbf{NTS}^s(\mathcal{S}, \mathcal{S}')$. *Then* $\mathsf{Flt}(\mu) \in \mathbf{NLS}^s(\mathcal{F}^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}'})$.

16.3.26. THEOREM. Flt *induces a functor from* $\mathbf{NTS}^s$ *to* $\mathbf{NLS}^s$.

16.3.27. DEFINITION. Let $\mathcal{D}$ be a λ-structure. Then $\mathsf{Comp}(\mathcal{D}) = (\mathcal{K}_s(\mathcal{D}), \leq)$ $\in \mathbf{MSL}^s$ can be turned into a strict intersection type structure by defining for $d, e \in \mathcal{K}_s(\mathcal{D})$

$$d \rightsquigarrow e = G(d \Rightarrow e).$$

Note that indeed $d \Rightarrow e$ is a strict stepfunction, as $d \neq \perp_{\mathcal{D}}$. Lemma 16.3.21(i) states that $d \rightsquigarrow e \in \mathcal{K}_s(\mathcal{D})$.

16.3.28. PROPOSITION. *Let* $\mathcal{D} \in \mathbf{NLS}^s$. *Then* $\mathcal{K}_s(\mathcal{D}) \in \mathbf{NTS}^s$.

16.3.29. PROPOSITION. *Let* $m \in \mathbf{NLS}^s(\mathcal{D}, \mathcal{D}')$. *Then* $\mathsf{Cmp}(m) \in \mathbf{NTS}^s(\mathcal{K}_s(\mathcal{D}), \mathcal{K}_s(\mathcal{D}'))$.

16.3.30. THEOREM. Cmp *induces a functor from* $\mathbf{NLS}^s$ *to* $\mathbf{NTS}^s$.

16.3.31. PROPOSITION. (i) *Let* $\mathcal{D} \in \mathbf{NLS}^s$. *Define* $m : \mathcal{F}^{\mathcal{K}_s(\mathcal{D})} \rightarrow \mathcal{D}$ *by*

$$m(X) = \bigsqcup X.$$

Then m is an $\mathbf{NLS}^s$ *isomorphism.*

(ii) *Let* $\mathcal{S} \in \mathbf{NTS}^s$. *Define* $\varphi : \mathcal{S} \rightarrow \mathcal{K}_s(\mathcal{F}^{\mathcal{S}})$ *by*

$$\varphi(s) = \uparrow s.$$

Then φ is an NTSs isomorphism.

16.3.32. THEOREM. *The categories* $\mathbf{NTS}^s$ *and* $\mathbf{NLS}^s$ *are equivalent. In fact the isomorphisms in 16.3.31 form natural isomorphisms showing*

$$\mathsf{Cmp} \circ \mathsf{Flt} \cong \mathbf{1}_{\mathbf{NTS}^s} \ \&\ \mathsf{Flt} \circ \mathsf{Cmp} \cong \mathbf{1}_{\mathbf{NLS}^s}.$$

16.3.33. LEMMA. *Let* $S \in \mathbf{NTS}_s$ *and let* $f \in [\mathcal{F}^{\mathcal{S}}_s \rightarrow \mathcal{F}^{\mathcal{S}}_s]$. *Then*

$$f \text{ is representable } \Leftrightarrow F^{\mathcal{S}} \circ G^{\mathcal{S}}(f) = f.$$

PROOF. As before. ■

16.3.34. THEOREM. *let* $* \in \mathbf{NTS}_s$. *Let $\mathcal{R}$ be the set of representable functions* $\mathcal{F}^{\mathcal{S}}_s \rightarrow \mathcal{F}^{\mathcal{S}}_s$. *Then we have the following.*

(i) *$\mathcal{R}$ contains the only constant strict function* $\mathsf{K}\perp_{\mathcal{F}^{\mathcal{S}}_s}$.

(ii) $\mathcal{R}$ *contains the continous strict step functions iff for all* $a, b, c, d \in \mathcal{S}$

$$a \to b \leq c \to d \;\Rightarrow\; c \leq a \;\&\; b \leq d.$$

(iii) $\mathcal{R}$ *contains the continous strict functions iff*

$$\begin{array}{lll} \forall n \geq 1, \forall a_1, \dots, a_n, b_1, \dots, b_n, c, d \in \mathcal{S} & & \\ (a_1 \to b_1) \cap \dots (a_n \to b_n) \leq (c \to d) & \Rightarrow & [c \leq a_{i_1} \cap \dots \cap a_{i_k} \;\&\; b_{i_1} \cap \dots \cap b_{i_k} \leq d] \\ & & \textit{for some } k \geq 1,\ 1 \leq i_1 < \dots < i_k \leq n. \end{array}$$

PROOF. (i) The map $\mathsf{K}\bot_{\mathcal{F}^{\mathcal{S}}_s}$ is clearly represented by $\uparrow\emptyset$.

(ii) ($\Rightarrow$) Suppose $a \to b \leq c \to d$. Take $h = \uparrow a \Rightarrow \uparrow b$. Then

$$\begin{array}{rcl} (F^{\mathcal{S}}_s \circ G^{\mathcal{S}}_s)(h)(\uparrow c) & = & \{e \mid a \to b \leq c \to e\} \\ h(\uparrow c) & = & \begin{cases} \uparrow b & \text{if } c \leq a; \\ \emptyset & \text{else} \end{cases} \end{array}$$

By the assumption we get $(F^{\mathcal{S}}_s \circ G^{\mathcal{S}}_s)(h)(\uparrow c) = h(\uparrow c)$. Therefore $c \leq a \;\&\; b \leq d$.

($\Leftarrow$) As before. By (i) we only need to consider functions $X \Rightarrow Y$ where $Y \neq \bot$.

(iii) The proof is similar to the proof of theorem 16.2.25(iv). Now the condition in the RHS of (iii) is equivalent with

$$(a_1 \to b_1) \cap \dots \cap (a_n \to b_n) \leq (c \to d) \;\Leftrightarrow\; \{i \mid c \leq a_i\} \neq \emptyset \;\&\; \bigcap\{b_i \mid c \leq a_i\} \leq d.$$

Let $h = (\uparrow a_1 \Rightarrow \uparrow b_1) \sqcup \dots \sqcup (\uparrow a_n \Rightarrow \uparrow b_n)$, with $n \geq 1$. Then

$$h(\uparrow c) = \begin{cases} \{d \mid \bigcap\{b_i \mid c \leq a_i\} \leq d\} & \text{if } \{i \mid c \leq a_i\} \neq \emptyset \\ \emptyset & \text{else} \end{cases}$$

and

$$(F^{\mathcal{S}} \circ G^{\mathcal{S}})(h)(\uparrow c) = \{d \mid (a_1 \to b_1) \cap \dots \cap (a_n \to b_n) \leq (c \to d)\}.$$

Note that by (i) $\mathsf{K}\bot$ is representable and that for other $h \in \mathcal{K}([\mathcal{F}^{\mathcal{S}}_s \to^s \mathcal{F}^{\mathcal{S}}_s])$ we have

$$h = (\uparrow a_1 \Rightarrow \uparrow b_1) \sqcup \dots \sqcup (\uparrow a_n \Rightarrow \uparrow b_n),$$

with $n \geq 1$. The rest of the proof is as in theorem 16.2.25(iv). ∎

16.4. Filter models 26.2.2004:1018

Given a natural λ-structure $\mathcal{D} = \langle \mathcal{D}, F, G \rangle$ we can interprete (untyped) λ-terms in it. For those $\mathcal{D}$ that are a filter model, the interpretation will have a simple form.

16.4.1. DEFINITION. (i) Let $\mathcal{D}$ be a λ-structure. An environment in $\mathcal{D}$ is a map $\rho : \text{Var} \to \mathcal{D}$. The set of environments in $\mathcal{D}$ is denoted by $\mathsf{Env}_{\mathcal{D}}$.

(ii) If $\rho \in \mathsf{Env}_{\mathcal{D}}$ and $d \in \mathcal{D}$, then $\rho[x := d]$ is the $\rho' \in \mathsf{Env}_{\mathcal{D}}$ defined by

$$\begin{array}{rcll} \rho'(x) & = & d; & \\ \rho'(y) & = & \rho(y), & \quad \text{if } y \neq x. \end{array}$$

16.4.2. Definition. Let $\langle \mathcal{D}, F, G \rangle$ be a λ-structure. The interpretation under ρ in $\mathcal{D}$ of an untyped λ-term M, notation $[\![M]\!]^{\mathcal{D}}_{\rho}$, is defined inductively as follows.

$$\begin{array}{rcl} [\![x]\!]^{\mathcal{D}}_{\rho} & = & \rho(x); \\ [\![MN]\!]^{\mathcal{D}}_{\rho} & = & F([\![M]\!]^{\mathcal{D}}_{\rho})([\![N]\!]^{\mathcal{D}}_{\rho}); \\ [\![\lambda x.M]\!]^{\mathcal{D}}_{\rho} & = & G(\boldsymbol{\lambda} d \in \mathcal{D}.[\![M]\!]^{\mathcal{D}}_{\rho[x:=d]}). \end{array}$$

Notice that the function $\boldsymbol{\lambda} d \in \mathcal{D}.[\![M]\!]^{\mathcal{D}}_{\rho[x:=d]}$ used for defining $[\![\lambda x.M]\!]^{\mathcal{D}}_{\rho}$ is continuous.

If $\mathcal{D} = \mathcal{F}^{\mathcal{S}}$, then we write $[\![M]\!]^{\mathcal{S}}_{\rho}$ for $[\![M]\!]^{\mathcal{D}}_{\rho}$.

An important result is that for filter structures the interpretation of a term coincides with the set of types which are deducible for it. For this we extend the notion of a context.

16.4.3. Definition. (i) A *flexible context* Γ is a set, now possibly infinite, of statements of the form $x : A$ where the requirement

$$(x{:}A), (x{:}B) \in \Gamma \;\Rightarrow\; A = B$$

is dropped.

(ii) If ρ is a valuation in a filter model $\mathcal{F}^{\nabla}$, then we define the flexible context $\Gamma_{\rho} = \{(x{:}B) \mid B \in \rho(x)\}$.

(iii) For a flexible context Γ we define

$$\Gamma \vdash M : A \;\Leftrightarrow\; \exists \Gamma_0 \subseteq \Gamma . \Gamma_0 \text{ finite } \;\&\; \cap \Gamma_0 \vdash M : A,$$

where $\cap \Gamma_0 = \{x{:}(B_1 \cap \ldots \cap B_k) \mid (x{:}B_i) \in \Gamma_0\}$.

16.4.4. Theorem. *Let $\mathcal{S} \in \mathbf{NTS}$ with $\langle \mathcal{F}^{\mathcal{S}}, F^{\mathcal{S}}, G^{\mathcal{S}} \rangle$ its corresponding filter structure. Then, for any λ-term M and $\rho \in \mathsf{Env}^{\mathcal{S}}_{\mathcal{F}}$,*

$$[\![M]\!]^{\mathcal{S}}_{\rho} = \{A \in \mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}} M : A\},$$

where $\Gamma_{\rho} = \{x{:}B \mid B \in \rho(x)\}$ is a flexible context.

Proof. By induction on the structure of M.
If $M \equiv x$, then

$$\begin{array}{rcll} [\![x]\!]^{\mathcal{S}}_{\rho} & = & \rho(x) & \\ & = & \{B \mid B \in \rho(x)\} & \\ & = & \{A \in \mathcal{S} \mid \exists B \in \rho(x).\ B \leq A\} & \\ & = & \{A \in \mathcal{S} \mid \exists B \in \rho(x).\ x{:}B \vdash^{\mathcal{S}} x : A\}, & \quad \text{by Theorem 15.1.16(i)} \\ & = & \{A \in \mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}} x : A\}. & \end{array}$$

If $M \equiv NL$, then

$$\begin{aligned}
[\![NL]\!]^{\mathcal{S}}_{\rho} &= [\![N]\!]^{\mathcal{S}}_{\rho} \cdot [\![L]\!]^{\mathcal{S}}_{\rho} \\
&= \{A{\in}\mathcal{S} \mid \exists B{\in}[\![L]\!]^{\mathcal{S}}_{\rho}.(B \to A){\in}[\![N]\!]^{\mathcal{S}}_{\rho}\} \\
&= \{A{\in}\mathcal{S} \mid \exists \vec{B}, \vec{C}.(B_i{\to}C_i){\in}[\![N]\!]^{\mathcal{S}}_{\rho} \ \&\ B_i{\in}[\![L]\!]^{\mathcal{S}}_{\rho} \ \&\ \bigcap_{i\in I} C_i \leq A\} \\
&= \{A{\in}\mathcal{S} \mid \exists \vec{B}, \vec{C}.\Gamma_{\rho} \vdash^{\mathcal{S}} N : (B_i{\to}C_i) \ \&\ \Gamma_{\rho} \vdash^{\mathcal{S}} L : B_i \ \&\ \bigcap \vec{C} \leq A\}, \\
&\quad\ \text{using the IH,} \\
&= \{A{\in}\mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}} NL : A\},
\end{aligned}$$

by Theorem 15.1.2(ii). If $M \equiv \lambda x.N$, then

$$\begin{aligned}
[\![\lambda x.N]\!]^{\mathcal{S}}_{\rho} &= G^{\mathcal{S}}(\boldsymbol{\lambda} X{\in}\mathcal{F}^{\mathcal{S}}.[\![N]\!]^{\mathcal{S}}_{\rho[x:=X]}) \\
&= \uparrow \{(B{\to}C){\in}\mathcal{S} \mid C{\in}[\![N]\!]^{\mathcal{S}}_{\rho[x:=\uparrow B]}\} \\
&= \{A{\in}\mathcal{S} \mid \exists \vec{B}, \vec{C}.\Gamma_{\rho} \vdash^{\mathcal{S}} N : C_i \ \&\ \bigcap_{i\in I} (B_i{\to}C_i) \leq A\}, \\
&\quad\ \text{by the IH,} \\
&= \{A{\in}\mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}} \lambda x.N : A\}
\end{aligned}$$

by Theorem 15.1.2(iii) and rule $(\leq)$. ■

Theorem 16.4.4 is also valid for **WNTS**s, as will be proved in Exercise 16.6.4.

16.4.5. Theorem. (i) *Let* $\mathcal{S}{\in}$**NTS** *and* $\mathcal{F}^{\mathcal{S}}$ *its corresponding filter structure. Then, for any* λ*-term* M *and* $\rho{\in}\mathsf{Env}_{\mathcal{F}^{\mathcal{S}}}$,

$$[\![M]\!]^{\mathcal{F}^{\mathcal{S}}}_{\rho} = \{A{\in}\mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}\Omega} M : A\},$$

where $\Gamma_{\rho} = \{x{:}B \mid B{\in}\rho(x)\}$ *is a flexible context.*

(ii) *Let* $\mathcal{S}{\in}\mathbf{NTS^s}$ *and* $\mathcal{F}^{\mathcal{S}}_s$ *its corresponding strict filter structure. Then, for any* λ*-term* M, $\rho{\in}\mathsf{Env}_{\mathcal{F}^{\mathcal{S}}_s}$ *and* Γ_{ρ} *as above,*

$$[\![M]\!]^{\mathcal{F}^{\mathcal{S}}_s}_{\rho} = \{A{\in}\mathcal{S} \mid \Gamma_{\rho} \vdash^{\mathcal{S}} M : A\}.$$

%Move to 16.4.4%

16.4.6. Proposition. (i) *Let* $\mathcal{T}$ *be an* **NTS** *that is* β*-sound. Then* $\mathcal{F}^{\mathcal{T}}$ *is a* λ*-model.*

(ii) *Let* $\mathcal{T}$ *be an* $\mathbf{NTS}^s$ *that is* β*-sound. Suppose (in case there is a top* Ω*)*

$$\Gamma \vdash^{\mathcal{T}} (\lambda x.M) : (B{\to}\Omega) \ \Rightarrow\ \Gamma, x{:}B \vdash^{\mathcal{T}} M : \Omega.$$

Then $\mathcal{F}^{\mathcal{T}}_s$ *is a* $\lambda\mathsf{I}$*-model.*

PROOF. (i) The only thing that remains to verify is

$$[\![(\lambda x.M)N]\!]_\rho = [\![M[x{:}=N]]\!]_\rho. \qquad \text{(beta)}$$

This follows from Theorem 16.4.5(i).
(ii) This time (beta) follows from Theorems 16.4.5(ii) and 15.2.9(i). ∎

16.4.7. COROLLARY. (i) *Let* $\nabla \in \{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{BCD}, \mathit{HL}\}$. *Then*

$$\mathcal{F}^\nabla \textit{ is a } \lambda\textit{-model.}$$

(ii) *Let* $\nabla \in \{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{BCD}, \mathit{HL}, \mathit{CDV}\}$. *Then*

$$\mathcal{F}_s^\nabla \textit{ is a } \lambda\mathsf{I}\textit{-model.}$$

PROOF. (i) By (i) of the Proposition using Corollary 15.1.19.
(ii) By (ii) of the Proposition using Corollary 15.1.19. ∎

16.4.8. PROPOSITION. (i) *Let* $\mathcal{T}$ *be an* **NTS** *that is* β*- and* $\eta\Omega$*-sound. Then*

$$\mathcal{F}^\mathcal{T} \textit{ is an extensional } \lambda\textit{-model.}$$

(ii) *Let* $\mathcal{T}$ *be an* $\mathbf{NTS}^s$ *that is* β*- and* η*-sound. Suppose (in case there is a top* Ω*)*

$$\Gamma \vdash^\mathcal{T} (\lambda x.M) : (B \to \Omega) \;\Rightarrow\; \Gamma, x{:}B \vdash^\mathcal{T} M : \Omega.$$

Then

$$\mathcal{F}_s^\mathcal{T} \textit{ is an extensional } \lambda\mathsf{I}\textit{-model.}$$

PROOF. (i) Now one needs to verify for $x \notin \mathrm{FV}(M)$

$$[\![\lambda x.Mx]\!]_\rho = [\![M]\!]_\rho. \qquad \text{(eta)}$$

This follows from Theorems 16.4.5(i), 16.4.6(i) and **??** as $\mathcal{T}$ is natural.
(ii) This time (eta) follows from Theorems 16.4.5(ii), 16.4.6(ii) and **??**. ∎

16.4.9. COROLLARY. (i) *Let* $\nabla \in \{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}\}$. *Then*

$$\mathcal{F}^\nabla \textit{ is an extensional } \lambda\textit{-model.}$$

(ii) *Let* $\nabla \in \{\mathit{Scott}, \mathit{Park}, \mathit{CDZ}, \mathit{HR}, \mathit{DHM}, \mathit{HL}\}$. *Then*

$$\mathcal{F}_s^\nabla \textit{ is an extensional } \lambda\mathsf{I}\textit{-model.}$$

PROOF. (i) By Proposition 15.2.14 and Theorem 15.1.12(ii), realizing that $\mathcal{T}^{\mathrm{AO}}$ and $\mathcal{T}^{\mathrm{HL}}$ are not natural type structures.
(ii) By Propositions 15.2.14 and Theorems 15.1.12(ii) and 15.1.19. ∎

%%

Examples of filter λ-models

Theorem **??**(**??**) together with the Generation Theorems **??** and 15.1.16(**??**) yield immediately that the theories $\nabla^{\triangledown}$ for $\triangledown\in\{\mathrm{AO},\mathrm{BCD},\mathrm{Scott},\mathrm{Park},\mathrm{HL},\mathrm{HR}\}$ induce filter λ-models. All the above theories are Ω-natural and satisfy the conditions of Theorem **??**(**??**), and therefore by the same theorem, in all the filter models induced by such theories all continuous functions are representable. In Section 17.3 we will give many more filter λ-models arising from domain models.

It is interesting, therefore, to ask whether there exist filter λ-models where *not all* continuous functions are representable. We answer affirmatively, and end this section by giving two examples of type theories which are Ω-natural but do not satisfy the first condition of Theorem **??**(**??**) and therefore by the same theorem not all continuous functions are representable in the induced filter structures.

The first model builds on an idea in Coppo et al. [1984], the second derives from Alessi [1993].

The theory $\nabla^{\diamondsuit}$

16.4.10. Definition. Let the ITT $\diamondsuit$ with constants $\mathbb{C}^{\diamondsuit}=\{\Omega,\diamondsuit,\heartsuit\}$ be axiomatized by rule (η) and axioms $(\rightarrow\text{-}\cap),(\Omega),(\Omega\eta)$ and $\{(\diamondsuit)\}$ where

$$(\diamondsuit)\ A\leq_{\diamondsuit}A[\diamondsuit:=\heartsuit].$$

16.4.11. Lemma.

$$\begin{array}{lrcl}
\text{(i)} & A\leq_{\diamondsuit}B & \Rightarrow & A[\diamondsuit:=\heartsuit]\leq_{\diamondsuit}B[\diamondsuit:=\heartsuit].\\
\text{(ii)} & \Gamma\vdash^{\diamondsuit}M:A & \Rightarrow & \Gamma[\diamondsuit:=\heartsuit]\vdash^{\diamondsuit}M:A[\diamondsuit:=\heartsuit].\\
\text{(iii)} & \Gamma,\Gamma'\vdash^{\diamondsuit}M:A & \Rightarrow & \Gamma,\Gamma'[\diamondsuit:=\heartsuit]\vdash^{\diamondsuit}M:A[\diamondsuit:=\heartsuit].
\end{array}$$

(iv) $\forall i{\in}I.\,\Gamma,x{:}A_i\vdash^{\diamondsuit}M:B_i\ \&\ \bigcap_{i\in I}(A_i\rightarrow B_i)\leq_{\diamondsuit}\bigcap_{j\in J}(C_j\rightarrow D_j)$
$\Rightarrow\ \forall j{\in}J.\,\Gamma,x{:}C_j\vdash^{\diamondsuit}M:D_j.$

Proof. (i) By induction on the definition of $\leq_{\diamondsuit}$.

(ii) By induction on derivations using (i) for rule $(\leq_{\diamondsuit})$.

(iii) From (ii) and rule ($\leq_{\diamondsuit}$ L), taking into account that if $(x{:}B)\in\Gamma$, then $(x{:}B[\diamondsuit:=\heartsuit])\in\Gamma[\diamondsuit:=\heartsuit]$ and $B\leq_{\diamondsuit}B[\diamondsuit:=\heartsuit]$.

(iv) We will denote by ψ,ξ (possibly with indices) elements of $\mathbb{C}^{\diamondsuit}$. We show by induction on the definition of $\leq_{\diamondsuit}$ that

$$\Big[\Big(\bigcap_{i\in I}(A_i\rightarrow B_i)\cap(\bigcap_{h\in H}\psi_h)\Big)\leq_{\diamondsuit}\Big(\bigcap_{j\in J}(C_j\rightarrow D_j)\cap(\bigcap_{k\in K}\xi_k)\Big)$$
$$\&\ \forall i{\in}I.\,\Gamma,x{:}A_i\vdash^{\diamondsuit}M:B_i\Big]\ \Rightarrow\ \forall j{\in}J.\,\Gamma,x{:}C_j\vdash^{\diamondsuit}M:D_j.$$

The only interesting case is when the applied rule is $(\diamondsuit)$, i.e. we have

$$\Big(\bigcap_{i\in I}(A_i\rightarrow B_i)\cap(\bigcap_{h\in H}\psi_h)\Big)\leq_{\diamondsuit}\Big(\bigcap_{i\in I}(A_i\rightarrow B_i)\cap(\bigcap_{h\in H}\psi_h)\Big)[\diamondsuit:=\heartsuit].$$

By hypothesis $\Gamma,x{:}A_i\vdash^{\diamondsuit}M:B_i$, so we are done by (iii). ■

16.4.12. Theorem. (i) $\nabla^{\diamondsuit}$ *is not* β*-sound.*

(ii) *Nevertheless* $\mathcal{F}^{\diamondsuit}$ *is a filter* λ*-model.*

PROOF. (i) We have $\diamondsuit{\rightarrow}\diamondsuit \leq_{\diamondsuit} \heartsuit{\rightarrow}\heartsuit$, but $\heartsuit \not\leq_{\diamondsuit} \diamondsuit$.

(ii) To show that $\mathcal{F}^{\diamondsuit}$ is a λ-model, it suffices, by Theorem **??**, to verify that $\Gamma \vdash^{\diamondsuit} \lambda x.M : A{\rightarrow}B \;\Rightarrow\; \Gamma, x{:}A \vdash^{\diamondsuit} M : B$. By Lemma 15.1.16(ii) $\Gamma \vdash^{\diamondsuit} \lambda x.M : A{\rightarrow}B \;\Rightarrow\; \Gamma, x{:}C_i \vdash^{\diamondsuit} M : D_i$ for some I, C_i, D_i such that $\bigcap_{i\in I}(C_i{\rightarrow}D_i) \leq_{\diamondsuit} A{\rightarrow}B$. So, we are done by Lemma 16.4.11(iv). ∎

The step function $\uparrow\diamondsuit \Rightarrow \uparrow\diamondsuit$ is for example not representable in $\mathcal{F}^{\diamondsuit}$.

The theory $\clubsuit$

16.4.13. DEFINITION. Define $\mathtt{t} : \mathbb{T}(\{\Omega,\varphi,\omega\}){\rightarrow}\mathbb{T}(\{\Omega,\varphi,\omega\})$ inductively:

$$\begin{array}{rcll} \mathtt{t}(\psi) & = & \psi, & \text{where } \psi\in\{\Omega,\omega\};\\ \mathtt{t}(\varphi) & = & \Omega; & \\ \mathtt{t}(A{\rightarrow}B) & = & A{\rightarrow}\mathtt{t}(B); & \\ \mathtt{t}(A\cap B) & = & \mathtt{t}(A)\cap\mathtt{t}(B). & \end{array}$$

The intersection type theory $\clubsuit$ is axiomatized by rule (η) and axioms $({\rightarrow}\text{-}\cap)$, $(\Omega), (\Omega\eta), (\omega\varphi), (\varphi{\rightarrow}\omega), (\omega{\rightarrow}\varphi)$, see Fig. 14.2, and $(\clubsuit), (\clubsuit{\rightarrow})$, where

$$\begin{array}{ll} (\clubsuit) & A \leq \mathtt{t}(A).\\ (\clubsuit{\rightarrow}) & A{\rightarrow}B \leq \mathtt{t}(A){\rightarrow}\mathtt{t}(B). \end{array}$$

If $\Gamma = \{x_1{:}A_1,\dots,x_n{:}A_n\}$, then write $\mathtt{t}(\Gamma) = \{x_1{:}\mathtt{t}(A_1),\dots,x_n{:}\mathtt{t}(A_n)\}$.

16.4.14. LEMMA. (i) *The map* t *is idempotent, i.e.* $\mathtt{t}(\mathtt{t}(A)) = \mathtt{t}(A)$.

(ii) $A{\rightarrow}\mathtt{t}(B) \;\sim_{\clubsuit}\; \mathtt{t}(A){\rightarrow}\mathtt{t}(B)$.

(iii) $A \leq_{\clubsuit} B \;\Rightarrow\; \mathtt{t}(A) \leq_{\clubsuit} \mathtt{t}(B)$.

(iv) $\Gamma \vdash^{\clubsuit} M : A \;\Rightarrow\; \mathtt{t}(\Gamma) \vdash^{\clubsuit} M : \mathtt{t}(A)$.

(v) $\Gamma, \Gamma' \vdash^{\clubsuit} M : A \;\Rightarrow\; \Gamma, \mathtt{t}(\Gamma') \vdash^{\clubsuit} M : \mathtt{t}(A)$.

(vi) $\forall i{\in}I.\, \Gamma, x{:}A_i \vdash^{\clubsuit} M : B_i \;\&\; \bigcap_{i\in I}(A_i{\rightarrow}B_i) \leq_{\clubsuit} \bigcap_{j\in J}(C_j{\rightarrow}D_j) \;\Rightarrow$
$\forall j{\in}J.\, \Gamma, x{:}C_j \vdash^{\clubsuit} M : D_j$.

PROOF. (i) Easy.

(ii) We get $\mathtt{t}(A) \rightarrow \mathtt{t}(B) \leq_{\clubsuit} A \rightarrow \mathtt{t}(B)$ by axiom $(\clubsuit)$ and rule (η). Morever $A \rightarrow \mathtt{t}(B) \leq_{\clubsuit} \mathtt{t}(A) \rightarrow \mathtt{t}(\mathtt{t}(B)) = \mathtt{t}(A) \rightarrow \mathtt{t}(B)$ by axiom $(\clubsuit \rightarrow)$ and (i).

(iii) By induction on the definition of $\leq_{\clubsuit}$ using (i) and (ii).

(iv) By induction on derivations using (iii) for rule $(\leq_{\clubsuit})$. We give the details just for rule $({\rightarrow}\mathrm{E})$. Suppose $M \equiv NL$ and $\Gamma \vdash^{\clubsuit} N : B{\rightarrow}A$, $\Gamma \vdash^{\clubsuit} L : B$. Then, by induction, $\mathtt{t}(\Gamma) \vdash^{\clubsuit} N : \mathtt{t}(B{\rightarrow}A)$, $\mathtt{t}(\Gamma) \vdash^{\clubsuit} L : \mathtt{t}(B)$. Since $\mathtt{t}(B{\rightarrow}A) = B{\rightarrow}\mathtt{t}(A)$, by (ii) we get $\mathtt{t}(\Gamma) \vdash^{\clubsuit} NL : \mathtt{t}(A)$.

(v) From (iv) and rule $(\leq_{\clubsuit}\text{ L})$, taking into account that if $(x{:}B){\in}\Gamma$, then $(x{:}\mathtt{t}(B)){\in}\mathtt{t}(\Gamma)$ and $B \leq_{\clubsuit} \mathtt{t}(B)$.

(vi) We will denote by ψ, ξ (possibly with indices) elements of $\{\Omega, \varphi, \omega\}$. We show by induction on the definition of $\leq_{\clubsuit}$ that

$$(\bigcap_{i\in I}(A_i{\to}B_i)) \cap (\bigcap_{h\in H}\psi_h) \leq_{\clubsuit} (\bigcap_{j\in J}(C_j{\to}D_j)) \cap (\bigcap_{k\in K}\xi_k)$$

$$\text{and } \forall i{\in}I.\ \Gamma, x{:}A_i \vdash^{\clubsuit} M : B_i \text{ imply } \forall j{\in}J.\ \Gamma, x{:}C_j \vdash^{\clubsuit} M : D_j.$$

The only interesting case is when the applied rules are ($\clubsuit$) or ($\clubsuit\!\to$), i.e. we have $A \to B \leq_{\clubsuit} \mathtt{t}(A \to B) = A \to \mathtt{t}(B)$ or $A{\to}B \leq_{\clubsuit} \mathtt{t}(A){\to}\mathtt{t}(B)$. By hypothesis $\Gamma, x{:}A \vdash^{\clubsuit} M : B$, so we are done by (v). ∎

16.4.15. Theorem. (i) $\nabla^{\clubsuit}$ *is not a* β*-theory.*
(ii) $\mathcal{F}^{\clubsuit}$ *is a filter* λ*-model.*

Proof. (i) For instance, $\varphi{\to}\omega \leq_{\clubsuit} \Omega{\to}\omega$, but $\Omega \not\leq_{\clubsuit} \varphi$.
(ii) As in the proof of Theorem 16.4.12, Lemma 16.4.14(vi) allows to prove that the conditions of Theorem **??** hold, hence $\mathcal{F}^{\clubsuit}$ is a λ-model. ∎

The step function $\uparrow \varphi \Rightarrow \uparrow \omega$ is not representable in $\mathcal{F}^{\clubsuit}$.

Actually, $\mathcal{F}^{\clubsuit}$ is the inverse limit solution of the domain equation $\mathcal{D} \simeq [\mathcal{D}{\to}\mathcal{D}]$ taken in the category of $\clubsuit$*-lattices*, whose objects are ω-algebraic lattices $\mathcal{D}$ endowed with a finitary additive projection $\delta : \mathcal{D}{\to}\mathcal{D}$ and whose morphisms $f : (\mathcal{D}, \delta){\to}(\mathcal{D}', \delta')$ are continuous functions such that $\delta' \circ f \sqsubseteq f \circ \delta$, see Alessi [1993].

16.5. Approximation theorems

A crucial result in the study of the equational theory of ω-algebraic λ-models are the *Approximation Theorems*, see e.g. Hyland [1975/76], Wadsworth [1976], Barendregt [1984], Longo [1987], Ronchi Della Rocca [1988], Honsell and Ronchi Della Rocca [1992]. An Approximation Theorem expresses the interpretation of any λ-term, even a non terminating one, as the supremum of the interpretations of suitable *normal forms*, called the *approximants* of the term, in an appropriate *extended language*. Approximation Theorems are very useful in proving, for instance, *Computational Adequacy* of models with respect to *operational semantics*, see e.g. Barendregt [1984], Honsell and Ronchi Della Rocca [1992]. There are other possible methods of showing computational adequacy, both semantical and syntactical, e.g. Hyland [1975/76], Wadsworth [1976], Honsell and Ronchi Della Rocca [1992], Abramsky and Ong [1993], **?**, but the method based on Approximation Theorems is usually the most straightforward. However, proving an Approximation Theorem for a given model theory is usually rather difficult. Most of the proofs in the literature are based on the technique of *indexed reduction*, see Wadsworth [1976], Abramsky and Ong [1993], Honsell and Ronchi Della Rocca [1992]. However, when the model in question is a filter model, by applying duality, the Approximation Theorem can be rephrased as follows: the types of a given term are all and only the types of its approximants. This change in perspective opens the way to proving Approximation Theorems using the syntactical machinery of proof theory, such as *logical predicates* and *computability* techniques.

The aim of the present section is to show in a uniform way that all the type assignment systems which induce filter models isomorphic to the models in Abramsky and Ong [1993], Barendregt et al. [1983], Scott [1972], Park [1976], Coppo et al. [1987], Honsell and Ronchi Della Rocca [1992] satisfy the Approximation Theorem. To this end we use a technique which can be constructed as a version of stable sets over a Kripke applicative structure.

For almost all the type theories of Figure 17.3 which induce λ-models we introduce appropriate notions of *approximants* which agree with the λ-theories of different models and therefore also with the type theories describing these models. Then we will prove that all types of an approximant of a given term (with respect to the appropriate notion of approximants) are also types of the given term. Finally we show the converse, namely that the types which can be assigned to a term can also be assigned to at least one approximant of that term. Hence a type can be derived for a term *if and only if* it can be derived for an approximant of that term. We end this section showing some applications of the Approximation Theorem.

Approximate normal forms

In this section we consider two extensions of λ-calculus both obtained by adding one constant. The first one is the well known language $\lambda\bot$, see Barendregt [1984]. The other extension is obtained by adding Φ and it is discussed in Honsell and Ronchi Della Rocca [1992].

16.5.1. Definition. (i) The set $\Lambda\bot$ of $\lambda\bot$-terms is obtained by adding the constant $\bot$ to the formation rules of terms.

(ii) The set $\Lambda\Phi$ of $\lambda\Phi$-terms is obtained by adding the constant Φ to the formation rules of terms. ∎

We consider two mappings (α and β) from λ-terms to $\lambda\bot$-terms and one mapping (γ) from λ-terms to $\lambda\Phi$-terms. These mappings differ in the translation of β-redexes. Clearly the values of these mappings are β-irreducible terms, i.e. normal forms for an extended language. As usual we call such a term an *approximate normal form* or abbreviated an *anf.*

16.5.2. Definition. The mappings $\alpha : \Lambda\to\Lambda\bot, \beta : \Lambda\to\Lambda\bot, \gamma : \Lambda\to\Lambda\Phi$ are inductively defined as follows.

$$\begin{array}{rcl}
\dagger(\lambda\vec{x}.y\vec{M}) & = & \lambda\vec{x}.y \dagger (M_1)\ldots\dagger (M_m);\\
\alpha(\lambda\vec{x}.(\lambda y.R)N\vec{M}) & = & \bot;\\
\beta(\lambda\vec{x}.(\lambda y.R)N\vec{M}) & = & \lambda\vec{x}.\bot;\\
\gamma(\lambda\vec{x}.(\lambda y.R)N\vec{M}) & = & \lambda\vec{x}.\Phi\gamma(\lambda y.R)\gamma(N)\gamma(M_1)\ldots\gamma(M_m),
\end{array}$$

where $\dagger\in\{\alpha,\beta,\gamma\}$, $\vec{M}\equiv M_1\ldots M_m$ and $m\geq 0$. ∎

In order to give the appropriate Approximation Theorem we will use the mapping α for the type assignment systems $\lambda\cap^{\mathrm{BCD}}$, $\lambda\cap^{\mathrm{Scott}}$, $\lambda\cap^{\mathrm{CDZ}}$, $\lambda\cap^{\mathrm{DHM}}$ the mapping β for the type assignment system $\lambda\cap^{\mathrm{AO}}$, and the mapping γ for the type assignment systems $\lambda\cap^{\mathrm{Park}}$, $\lambda\cap^{\mathrm{HR}}$. Each one of the above mappings associates a set of approximants to each λ-term in the standard way.

16.5.3. Definition. Let $\bigtriangledown \in \{\text{BCD}, \text{Scott}, \text{CDZ}, \text{DHM}, \text{AO}, \text{Park}, \text{HR}\}$.
The set $\mathcal{A}_{\bigtriangledown}(M)$ of $\bigtriangledown$-*approximants* of M is defined by

$$\mathcal{A}_{\bigtriangledown}(M) = \{P \mid \exists M'.\ M \twoheadrightarrow_{\beta} M' \text{ and } P \equiv \dagger(M')\},$$

where

$$\begin{array}{lcll} \dagger & = & \alpha, & \text{for } \bigtriangledown \in \{\text{BCD}, \text{Scott}, \text{CDZ}, \text{DHM}\}, \\ \dagger & = & \beta, & \text{for } \bigtriangledown \in \{\text{AO}\}, \\ \dagger & = & \gamma, & \text{for } \bigtriangledown \in \{\text{Park}, \text{HR}\}. \blacksquare \end{array}$$

We extend the typing to $\lambda\perp$-terms and to $\lambda\Phi$-terms by adding two different axioms for Φ and nothing for $\perp$.

16.5.4. Definition. (i) Let $\bigtriangledown \in \{\text{BCD}, \text{Scott}, \text{CDZ}, \text{DHM}, \text{AO}\}$. We extend the definition of type assignment $\Gamma \vdash^{\bigtriangledown} M : A$ to $\lambda\perp$-terms by letting M, N in definition 14.2.2 range over $\Lambda\perp$.

(ii) We extend the type assignment $\lambda\cap^{\text{Park}}$ to $\lambda\Phi$-terms by adding the axiom (Ax-Φ-Park) $\quad \Gamma \vdash^{\text{Park}} \Phi : \omega$.

(iii) We extend the type assignment $\lambda\cap^{\text{HR}}$ to $\lambda\Phi$-terms by adding the axiom (Ax-Φ-HR) $\quad \Gamma \vdash^{\text{HR}} \Phi : \varphi$. $\blacksquare$

It is easy to verify that the appropriate generalization of the Generation Theorems (Theorems 15.1.2 and 15.1.16) holds also for these two extensions of the type assignment system. Therefore we do not introduce different notations for these extended type assignment systems and we will freely refer to the Generation Theorems in discussing derivations in these extended systems. We immediately get the following.

16.5.5. Proposition. (i) *Let $\bigtriangledown \in \{BCD, Scott, CDZ, DHM, AO\}$. Then*

$$\Gamma \vdash^{\bigtriangledown} \perp : A \ \Leftrightarrow\ A \sim_{\bigtriangledown} \Omega.$$

(ii) $\Gamma \vdash^{Park} \Phi : A \ \Leftrightarrow\ \omega \leq_{Park} A.$

(iii) $\Gamma \vdash^{HR} \Phi : A \ \Leftrightarrow\ \varphi \leq_{HR} A.$ $\blacksquare$

16.5.6. Lemma. (i) $M_1 \twoheadrightarrow_{\beta} M_2 \;\&\; \Gamma \vdash^{\bigtriangledown} \dagger(M_1) : A \ \Rightarrow\ \Gamma \vdash^{\bigtriangledown} \dagger(M_2) : A.$

(ii) *If $P, P' \in \mathcal{A}_{\bigtriangledown}(M)$, $\Gamma \vdash^{\bigtriangledown} P : A$ and $\Gamma \vdash^{\bigtriangledown} P' : B$, then*

$$\exists P'' \in \mathcal{A}_{\bigtriangledown}(M). \Gamma \vdash^{\bigtriangledown} P'' : A \cap B.$$

Proof. (i). For $\bigtriangledown \in \{\text{BCD}, \text{Scott}, \text{CDZ}, \text{DHM}, \text{AO}\}$ the proof is standard.
For $\bigtriangledown \in \{\text{Park}, \text{HR}\}$ it sufficies to consider the case $M_1 \equiv (\lambda x.M)N$ and $M_2 \equiv M[x := N]$. Notice that $\gamma(M[x := N])$ is $\gamma(M)$ where the occurrences of x have been replaced by $\Phi\gamma(N)$ if they are functional and N is an abstraction, and by $\gamma(N)$ otherwise. More formally if we define the mapping $\overline{\gamma} : \Lambda \to \Lambda\Phi$ by

$$\overline{\gamma}(M) = \begin{cases} \Phi\gamma(M) & \text{if } M \equiv \lambda x.M' \\ \gamma(M) & \text{otherwise} \end{cases}$$

and the mapping $\{\ \}^x_y : \Lambda \to \Lambda$ by

$$\begin{array}{lcl} \{z\}^x_y & = & z \\ \{M_1M_2\}^x_y & = & \begin{cases} y\{M_2\}^x_y & \text{if } M_1 \equiv x \\ \{M_1\}^x_y\{M_2\}^x_y & \text{otherwise} \end{cases} \\ \{\lambda z.M\}^x_y & = & \lambda z.\{M\}^x_y. \end{array}$$

then $\gamma(M_1M_2) = \overline{\gamma}(M_1)\gamma(M_2)$ and it is easy to check, by induction on M, that $\gamma(M[x := N]) \equiv \gamma(\{M\}^x_y)[x := \gamma(N)][y := \overline{\gamma}(N)]$ for y fresh.

We may assume $A \not\sim_{\nabla} \Omega$. Then from $\Gamma \vdash^{\nabla} \Phi(\lambda x.\gamma(M))\gamma(N) : A$ we get $\Gamma \vdash^{\nabla} \Phi(\lambda x.\gamma(M)) : C \to A$, $\Gamma \vdash^{\nabla} \gamma(N) : C$ for some C, by Theorem 15.1.16(ii). By Lemma 14.1.10(iv) one has $A \not\sim_{\nabla} \Omega$ we have $C \to A \not\sim_{\nabla} \Omega$, hence again by 15.1.16(ii) $\Gamma \vdash^{\nabla} \Phi : B \to C \to A$, $\Gamma \vdash^{\nabla} \lambda x.\gamma(M) : B$, for some B.

For ∇ = Park we get $\omega \leq_{\text{Park}} B \to C \to A$ from $\Gamma \vdash^{\text{Park}} \Phi : B \to C \to A$ by Proposition 16.5.5(ii). This implies $B \leq_{\text{Park}} \omega$, $C \leq_{\text{Park}} \omega$, and $\omega \leq_{\text{Park}} A$, since $\omega \sim_{\text{Park}} \omega \to \omega$, ∇^{Park} is β-sound by Theorem 15.1.12(ii), $C \to A \not\sim \Omega$ and $A \not\sim \Omega$. We obtain by rule $(\leq_{\text{Park}})$ $\Gamma \vdash^{\text{Park}} \lambda x.\gamma(M) : \omega$ and $\Gamma \vdash^{\text{Park}} \gamma(N) : \omega$. We get $\Gamma, x{:}\omega \vdash^{\text{Park}} \gamma(M) : \omega$ (by Theorem 15.1.16(ii)) and $\Gamma \vdash^{\text{Park}} \Phi\gamma(N) : \omega$ since $\omega \sim_{\text{Park}} \omega \to \omega$. Now $\gamma(\{M\}^x_y) = \gamma(M)$ with some occurrences of x replaced by the fresh variable y. Hence $\Gamma, y{:}\omega, x{:}\omega \vdash^{\text{Park}} \gamma(\{M\}^x_y){:}\omega$. So we conclude $\Gamma \vdash^{\text{Park}} \gamma(\{M\}^x_y)[x := \gamma(N)][y := \overline{\gamma}(N)] : A$ by rules (*cut*) and $(\leq_{\text{Park}})$.

For ∇ = HR we get $\varphi \leq_{\text{HR}} B \to C \to A$ from $\Gamma \vdash^{\text{HR}} \Phi : B \to C \to A$ by Theorem 16.5.5. This implies either ($B \leq_{\text{HR}} \varphi$ and $\varphi \leq_{\text{HR}} C \to A$) or ($B \leq_{\text{HR}} \omega$ and $\omega \leq_{\text{HR}} C \to A$) since $\varphi \sim_{\text{HR}} (\varphi \to \varphi) \cap (\omega \to \omega)$ and ∇^{HR} is β-sound by Theorem 15.1.12(ii), $C \to A \not\sim \Omega$ and $A \not\sim \Omega$ (notice that $\varphi \cap \omega \sim_{\text{HR}} \omega$). Similarly in the first case from $\varphi \leq_{\text{HR}} C \to A$ we get either $C \leq_{\text{HR}} \varphi$ and $\varphi \leq_{\text{HR}} A$ or $C \leq_{\text{HR}} \omega$ and $\omega \leq_{\text{HR}} A$. In the second case from $\omega \leq_{\text{HR}} C \to A$ we get $C \leq_{\text{HR}} \varphi$ and $\omega \leq_{\text{HR}} A$ since $\omega \sim_{\text{HR}} \varphi \to \omega$.

To sum up, using rule $(\leq_{\text{HR}})$ we have the following alternative cases.

- $\Gamma \vdash^{\text{HR}} \lambda x.\gamma(M) : \varphi$, $\Gamma \vdash^{\text{HR}} \gamma(N) : \varphi$, and $\varphi \leq_{\text{HR}} A$;
- $\Gamma \vdash^{\text{HR}} \lambda x.\gamma(M) : \varphi$, $\Gamma \vdash^{\text{HR}} \gamma(N) : \omega$, and $\omega \leq_{\text{HR}} A$;
- $\Gamma \vdash^{\text{HR}} \lambda x.\gamma(M) : \omega$, $\Gamma \vdash^{\text{HR}} \gamma(N) : \varphi$, and $\omega \leq_{\text{HR}} A$.

Therefore we get alternatively:

- $\Gamma, x{:}\varphi \vdash^{\text{HR}} \gamma(M) : \varphi$, and $\Gamma \vdash^{\text{HR}} \Phi\gamma(N) : \varphi$;
- $\Gamma, x{:}\omega \vdash^{\text{HR}} \gamma(M) : \omega$, and $\Gamma \vdash^{\text{HR}} \Phi\gamma(N) : \omega$;
- $\Gamma, x{:}\varphi \vdash^{\text{HR}} \gamma(M) : \omega$, and $\Gamma \vdash^{\text{HR}} \Phi\gamma(N) : \varphi$,

so we can conclude as in the previous case.

(ii). By hypotheses there are M_1, M_2 such that $M \twoheadrightarrow_\beta M_1$, $M \twoheadrightarrow_\beta M_2$ and $P \equiv \dagger(M_1)$, $P' \equiv \dagger(M_2)$. By the Church-Rosser property of $\to_\beta$ we can find M_3 such that $M_1 \twoheadrightarrow_\beta M_3$ and $M_2 \twoheadrightarrow_\beta M_3$. By (i) we can choose $P'' \equiv \dagger(M_3)$. ∎

Approximation Theorem - Part 1

It is useful to introduce the following definition.

16.5.7. Definition. Let $\triangledown \in \{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{AO}, \mathrm{Park}, \mathrm{HR}\}$. Write

$$[A]^{\triangledown}_{\Gamma} = \{M \mid \exists P \in \mathcal{A}_{\triangledown}(M).\ \Gamma \vdash^{\triangledown} P : A\}.$$

By definition we get that $M \in [A]^{\triangledown}_{\Gamma}$ and $N \twoheadrightarrow_{\beta} M$ imply $N \in [A]^{\triangledown}_{\Gamma}$. Moreover $\Gamma \Subset \Gamma'$ implies $[A]^{\triangledown}_{\Gamma} \subseteq [A]^{\triangledown}_{\Gamma'}$ for all types $A \in \mathbb{T}^{\triangledown}$.

In this subsection we prove that if $M : [A]^{\triangledown}_{\Gamma}$ then there exists a derivation of $\Gamma \vdash^{\triangledown} M : A$.

16.5.8. Proposition. *Let* $\triangledown \in \{BCD, Scott, CDZ, DHM, AO, Park, HR\}$.

$$M \in [A]^{\triangledown}_{\Gamma} \Rightarrow \Gamma \vdash^{\triangledown} M : A.$$

Proof. By Corollary 15.2.8(**??**) it is sufficient to show

$$\Gamma \vdash^{\triangledown} \dagger(M) : A \Rightarrow \Gamma \vdash^{\triangledown} M : A. \qquad (*)$$

where $\dagger = \alpha$ for $\nabla \in \{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{DHM}\}$,
$\dagger = \beta$ for $\triangledown = \mathrm{AO}$,
$\dagger = \gamma$ for $\triangledown \in \{\mathrm{Park}, \mathrm{HR}\}$.

For $\triangledown \in \{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{AO}\}$ from $\Gamma \vdash^{\triangledown} P : A$ we get $\Gamma \vdash^{\triangledown} M : A$ by Proposition 16.5.5(i) and the definition of the mappings α and β.
For $\triangledown \in \{\mathrm{Park}, \mathrm{HR}\}$ we prove $(*)$ by induction on M, assuming $A \not\sim_{\triangledown} \Omega$.

Case $M \equiv x$ is trivial.

Case $M \equiv \lambda x.M'$, then $P \equiv \lambda x.P'$ where $P' \equiv \gamma(M')$. By Theorem 15.1.2(**??**) from $\Gamma \vdash^{\triangledown} P : A$ we get $\Gamma, x{:}B_i \vdash^{\triangledown} P' : C_i$ and $\bigcap_{i \in I}(B_i \to C_i) \leq_{\triangledown} A$ for some I, B_i, C_i. We get by induction $\Gamma, x{:}B_i \vdash^{\triangledown} M' : C_i$ and so we conclude $\Gamma \vdash^{\triangledown} M : A$ using rules $(\to\mathrm{I})$, $(\cap\mathrm{I})$ and $(\leq_{\triangledown})$.

Case $M \equiv M_1 M_2$ where M_1 is not an abstraction. Then $P \equiv P_1 P_2$ where $P_1 \equiv \gamma(M_1)$ and $P_2 \equiv \gamma(M_2)$. By Theorem 15.1.16(**??**) from $\Gamma \vdash^{\triangledown} P : A$ we get $\Gamma \vdash^{\triangledown} P_1 : B \to A$, $\Gamma \vdash^{\triangledown} P_2 : B$ for some B. By induction this implies $\Gamma \vdash^{\triangledown} M_1 : B \to A$ and $\Gamma \vdash^{\triangledown} M_2 : B$, hence $\Gamma \vdash^{\triangledown} M \equiv M_1 M_2 : A$.

Case $M \equiv M_1 M_2$ where M_1 is an abstraction. Then $P \equiv \Phi P_1 P_2$ where $P_1 \equiv \gamma(M_1)$ and $P_2 \equiv \gamma(M_2)$. As in the proof of Lemma 16.5.6(i) from $\Gamma \vdash^{\triangledown} P : A$, where $A \not\sim_{\triangledown} \Omega$, we get $\Gamma \vdash^{\triangledown} \Phi : B \to C \to A$, $\Gamma \vdash^{\triangledown} P_1 : B$, $\Gamma \vdash^{\triangledown} P_2 : C$ for some B, C. By induction this implies $\Gamma \vdash^{\triangledown} M_1 : B$ and $\Gamma \vdash^{\triangledown} M_2 : C$.

For $\triangledown = \mathrm{Park}$, as in the proof of Lemma 16.5.6(i), we get $\Gamma \vdash^{\mathrm{Park}} M_1 : \omega$ and $\Gamma \vdash^{\mathrm{Park}} M_2 : \omega$. We can conclude $\Gamma \vdash^{\triangledown} M : A$ using rules $(\leq_{\mathrm{Park}})$ and $(\to\mathrm{E})$ since $\omega \sim_{\mathrm{Park}} \omega \to \omega$.

For $\triangledown = \mathrm{HR}$ as in the proof of Lemma 16.5.6(i) we have the following alternative cases.

- $\Gamma \vdash^{\mathrm{HR}} M_1 : \varphi$, $\Gamma \vdash^{\mathrm{HR}} M_2 : \varphi$, and $\varphi \leq_{\mathrm{HR}} A$;
- $\Gamma \vdash^{\mathrm{HR}} M_1 : \varphi$, $\Gamma \vdash^{\mathrm{HR}} M_2 : \omega$, and $\omega \leq_{\mathrm{HR}} A$;

- $\Gamma \vdash^{\mathrm{HR}} M_1 : \omega$, $\Gamma \vdash^{\mathrm{HR}} M_2 : \varphi$, and $\omega \leq_{\mathrm{HR}} A$.

It is easy to verify that in all cases we can derive $\Gamma \vdash^{\nabla} M : A$ from (I) and $(\varphi{\to}\omega)$ using rules $(\leq_{\mathrm{HR}})$ and $(\to\mathrm{E})$. ■

Approximation Theorem - Part 2

In order to prove the converse of Proposition 16.5.8 we will use a Kripke-like version of stable sets Mitchell [1996]. First we need a technical result.

16.5.9. Lemma. *Let $\triangledown{\in}\{BCD, Scott, CDZ, DHM, AO, Park, HR\}$. If $\Gamma' = \Gamma, z : B$, $z \notin \mathrm{FV}(M)$, and $A \not\sim_{\triangledown} \Omega$ for $\triangledown = AO$, then $Mz{\in}[A]^{\triangledown}_{\Gamma'}$ implies $M{\in}[B \to A]^{\triangledown}_{\Gamma}$.*

Proof. Let $P{\in}\mathcal{A}_{\triangledown}(Mz)$ and $\Gamma' \vdash^{\nabla} P : A$. We show by cases on P and M that there is $\hat{P}{\in}\mathcal{A}_{\triangledown}(M)$ such that $\Gamma \vdash^{\nabla} \hat{P} : B \to A$.

There are two possibilities.

- $Mz \twoheadrightarrow_{\beta} M'z$ and $P \equiv \dagger(M'z)$;
- $Mz \twoheadrightarrow_{\beta} (\lambda x.M')z \to_{\beta} M'[x := z]$ and $P{\in}\mathcal{A}_{\triangledown}(M'[x := z])$.

In the first case again there are two possibilities.

- $M' \equiv yM_1 \ldots M_m, \quad m \geq 0$;
- $M' \equiv (\lambda y.M_0)M_1 \ldots M_m, \quad m \geq 0$.

In total there are 4 cases:

- $P \equiv P'z$ and $P'{\in}\mathcal{A}_{\triangledown}(M)$;
- $P \equiv \bot$ and $\triangledown{\in}\{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{AO}\}$;
- $P \equiv \Phi P'z$, $P'{\in}\mathcal{A}_{\triangledown}(M)$ and $\triangledown{\in}\{\mathrm{Park}, \mathrm{HR}\}$;
- $M \twoheadrightarrow_{\beta} \lambda x.M'$ and $P{\in}\mathcal{A}_{\triangledown}(M'[x := z])$.

Case $P \equiv P'z$ where $P'{\in}\mathcal{A}_{\triangledown}(M)$ then we can choose $\hat{P} \equiv P'$. This is clear if $A \sim_{\triangledown} \Omega$ because then $\triangledown \neq \mathrm{AO}$, hence we have $(\Omega{-}\eta)$. Now let $A \not\sim_{\triangledown} \Omega$. Then by Theorem 15.1.16(**??**) from $\Gamma' \vdash^{\nabla} P : A$ we get $\Gamma' \vdash^{\nabla} P' : C \to A$, $\Gamma' \vdash^{\nabla} z : C$ for some C. $B \leq_{\triangledown} C$ by Theorem 15.1.16(**??**) and we conclude using $(\leq_{\triangledown})$ and (*strenghtening*) $\Gamma \vdash^{\nabla} P' : B \to A$.

Case $P \equiv \bot$ is trivial for $\triangledown{\in}\{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{DHM}\}$, because of $(\Omega{-}\eta)$ and impossible for $\triangledown = \mathrm{AO}$ by Proposition 16.5.5(i).

Case $P \equiv \Phi P'z$ where $P'{\in}\mathcal{A}_{\triangledown}(M)$ and $\triangledown{\in}\{\mathrm{Park}, \mathrm{HR}\}$, then we show that we can choose $\hat{P} \equiv P'$. Again let $A \not\sim_{\triangledown} \Omega$. Then from $\Gamma' \vdash^{\nabla} P : A$ we get by Theorem 15.1.16(ii) and (i) $\Gamma' \vdash^{\nabla} \Phi : C \to D \to A$, $\Gamma' \vdash^{\nabla} P' : C$, $\Gamma' \vdash^{\nabla} z : D$ for some C, D with $B \leq_{\triangledown} D$. For $\triangledown = \mathrm{Park}$ using Proposition 16.5.5(ii) as in the proof of Lemma 16.5.6(i), we get $C \leq_{\mathrm{Park}} \omega$, $D \leq_{\mathrm{Park}} \omega$, and $\omega \leq_{\text{Park}} A$ (remember that $\omega \sim_{\text{Park}} \omega{\to}\omega$). Similarly for $\bigtriangledown = \text{HR}$, using Proposition 16.5.5(iii), we get either $C \leq_{\text{HR}} \varphi$, $D \leq_{\text{HR}} \varphi$, and $\varphi \leq_{\text{Park}} A$ or $C \leq_{\text{HR}} \varphi$, $D \leq_{\text{HR}} \omega$, and $\omega \leq_{\text{HR}} A$ or $C \leq_{\text{Park}} \omega$, $D \leq_{\text{HR}} \varphi$, and $\omega \leq_{\text{Park}} A$ (remember that $\varphi \sim_{\text{HR}} (\varphi{\to}\varphi) \cap (\omega{\to}\omega)$ and $\omega \sim_{\text{HR}} \varphi{\to}\omega$). In all cases we can conclude $C \leq D \to A \leq B \to A$ and therefore by $(\leq)$ and (*strenghtening*) $\Gamma \vdash^{\nabla} P' : B \to A$.

Case $M \twoheadrightarrow_\beta \lambda x.M'$ and $P{\in}\mathcal{A}_{\bigtriangledown}(M'[x := z])$.If $\dagger = \alpha$ and $P \equiv \bot$ then we choose $\hat{P} \equiv P$. Else we choose $\hat{P} \equiv \lambda z.P$. ■

The following crucial definition is somewhat involved. It amounts essentially to the definition of the natural set-theoretic semantics of intersection types over a suitable Kripke applicative structure, where bases play the role of worlds.[2] In order to keep the treatment elementary we don't develop the full theory of the natural semantics of intersection types in Kripke applicative structures. The definition below is rather long, since we have different cases for the type ω and for arrow types according to the different type theories under consideration.

16.5.10. Definition (Kripke type interpretation).

Let $\bigtriangledown{\in}\{\text{BCD}, \text{Scott}, \text{CDZ}, \text{DHM}, \text{AO}, \text{Park}, \text{HR}\}$. Define

$$
\begin{array}{rcll}
[\![\psi]\!]_\Gamma^\bigtriangledown & = & [\psi]_\Gamma^\bigtriangledown & \text{for } \psi{\in}\mathbb{C}_\infty \cup \{\Omega, \varphi\} \\
 & & & \text{or } \psi = \omega \text{ and } \bigtriangledown = \text{Park}; \\
[\![\omega]\!]_\Gamma^\bigtriangledown & = & \{M \mid \forall \Gamma' \Supset \Gamma\, \forall \vec{N}{\in}[\varphi]_{\Gamma'}^\bigtriangledown . M\vec{N}{\in}[\omega]_{\Gamma'}^\bigtriangledown\} & \text{for } \bigtriangledown{\in}\{\text{CDZ}, \text{HR}\} \\
[\![\omega]\!]_\Gamma^\bigtriangledown & = & \{M \mid \forall \vec{N}. M\vec{N}{\in}[\omega]_\Gamma^\bigtriangledown\} & \text{for } \bigtriangledown{\in}\{\text{DHM}, \text{Scott}\}; \\
[\![A{\to}B]\!]_\Gamma^\bigtriangledown & = & \{M \mid \forall \Gamma' \Supset \Gamma\, \forall N{\in}[\![A]\!]_{\Gamma'}^\bigtriangledown . MN{\in}[\![B]\!]_{\Gamma'}^\bigtriangledown\}, & \text{if } \bigtriangledown \neq \text{AO or } B \not\sim_{\text{AO}} \Omega; \\
[\![A{\to}B]\!]_\Gamma^{\text{AO}} & = & [A{\to}B]_\Gamma^{\text{AO}}, & \text{if } B \sim_{\text{AO}} \Omega; \\
[\![A \cap B]\!]_\Gamma^\bigtriangledown & = & [\![A]\!]_\Gamma^\bigtriangledown \cap [\![B]\!]_\Gamma^\bigtriangledown . \blacksquare &
\end{array}
$$

If we dropped in clause (**??**) the condition $\bigtriangledown \neq$ AO or $B \not\sim_{\text{AO}} \Omega$ and we dropped the whole clause (**??**) then Lemma 16.5.11(ii) would not hold for $\bigtriangledown =$ AO.

Notice that, since $M{\in}[A]_\Gamma^\bigtriangledown$ and $N \twoheadrightarrow_\beta M$ imply $N{\in}[A]_\Gamma^\bigtriangledown$, the same property holds for $[\![A]\!]_\Gamma^\bigtriangledown$. Moreover $\Gamma \Subset \Gamma'$ implies $[\![A]\!]_\Gamma^\bigtriangledown \subseteq [\![A]\!]_{\Gamma'}^\bigtriangledown$ for all types $A{\in}\mathbb{T}^\bigtriangledown$.

The Lemmas 16.5.11, 16.5.14 and the final theorem are standard.

16.5.11. Lemma. *Let* $\bigtriangledown{\in}\{BCD, Scott, CDZ, DHM, AO, Park, HR\}$. *Then*

(i) $x\vec{M}{\in}[A]_\Gamma^\bigtriangledown \;\Rightarrow\; x\vec{M}{\in}[\![A]\!]_\Gamma^\bigtriangledown$;

(ii) $[\![A]\!]_\Gamma^\bigtriangledown \subseteq [A]_\Gamma^\bigtriangledown$.

Proof. (i) and (ii) can be simultaneously proved by induction on A. We consider only some interesting cases.

(i) Case $A \equiv \omega$ and $\bigtriangledown = \text{CDZ}$. Let $\Gamma' \Supset \Gamma$ and $\vec{N}{\in}[\varphi]_{\Gamma'}^{\text{CDZ}}$.

[2] As already observed in **?**) we cannot use here stable sets as we did in previous section since we need to take into account also the $\bigtriangledown$-bases, not only the λ-terms and their types.

Clearly $P\in\mathcal{A}_\nabla(x\vec{M}),\ \vec{Q}\in\mathcal{A}_\nabla(\vec{N}) \Rightarrow P\vec{Q}\in\mathcal{A}_\nabla(x\vec{M}\vec{N})$. Hence

$$\begin{array}{rcll} x\vec{M}\in[\omega]^{\mathrm{CDZ}}_\Gamma & \Rightarrow & x\vec{M}\vec{N}\in[\omega]^{\mathrm{CDZ}}_{\Gamma'} & \text{by rules } (\leq_{\mathrm{CDZ}}) \text{ and } (\to\mathrm{E}) \\ & & & \text{since } \omega \sim_{\mathrm{CDZ}} \varphi \to \omega, \\ & \Rightarrow & x\vec{M}\in[\![\omega]\!]^{\mathrm{CDZ}}_\Gamma & \text{by Definition 16.5.10(??).} \end{array}$$

Case $A \equiv B \to C$. Let $\Gamma' \Supseteq \Gamma$ and $\nabla \neq \mathrm{AO}$ or $C \not\sim_{\mathrm{AO}} \Omega$ and let $N\in[\![B]\!]^\nabla_{\Gamma'}$. $[\![B]\!]^\nabla_{\Gamma'} \subseteq [B]^\nabla_{\Gamma'}$ by induction on (ii). Hence

$$\begin{array}{rcll} x\vec{M}\in[A]^\nabla_\Gamma & \Rightarrow & x\vec{M}N\in[C]^\nabla_{\Gamma'} & \text{by rule } (\to\mathrm{E}), \\ & \Rightarrow & x\vec{M}N\in[\![C]\!]^\nabla_{\Gamma'} & \text{by induction on (i),} \\ & \Rightarrow & x\vec{M}\in[\![B \to C]\!]^\nabla_\Gamma & \text{by Definition 16.5.10(??).} \end{array}$$

(ii) Case $A \equiv B \to C$ and $\nabla \neq \mathrm{AO}$ or $C \not\sim_{\mathrm{AO}} \Omega$. Let $\Gamma' = \Gamma, z{:}B$ whith z fresh, and suppose $M\in[\![B \to C]\!]^\nabla_\Gamma$; as $z\in[\![B]\!]^\nabla_{\Gamma,z:B}$ by induction on (i), we have

$$\begin{array}{rcll} M\in[\![B \to C]\!]^\nabla_\Gamma \text{ and } z\in[\![B]\!]^\nabla_{\Gamma,z:B} & \Rightarrow & Mz\in[\![C]\!]^\nabla_{\Gamma'} & \text{by Definition 16.5.10(??),} \\ & \Rightarrow & Mz\in[C]^\nabla_{\Gamma'} & \text{by induction on (ii),} \\ & \Rightarrow & M\in[B\to C]^\nabla_\Gamma & \text{by Lemma 16.5.9.} \end{array}$$

Case $A \equiv B \cap C$. This follows from Lemma 16.5.6(ii). ■

The following lemma essentially states that the Kripke type interpretations agree with the corresponding type theories.

16.5.12. Lemma. (i) *Let* $\nabla\in\{\mathit{CDZ}, \mathit{DHM}\}$. *Then*

$$M\in[A]^\nabla_{\Gamma,z:\omega} \ \&\ N\in[\![\omega]\!]^\nabla_\Gamma \ \Rightarrow\ M[z := N]\in[A]^\nabla_\Gamma.$$

(ii) *Let* $\nabla\in\{\mathit{BCD}, \mathit{Scott}, \mathit{CDZ}, \mathit{DHM}, \mathit{AO}, \mathit{Park}, \mathit{HR}\}$. *Then*

$$\forall A, B\in\mathbb{T}^\nabla[A \leq_\nabla B \ \Rightarrow\ [\![A]\!]^\nabla_\Gamma \subseteq [\![B]\!]^\nabla_\Gamma].$$

Proof. (i) We may assume $A \not\sim_\nabla \Omega$. For $\nabla = \mathrm{CDZ}$ if $M\in[A]^{\mathrm{CDZ}}_{\Gamma,z:\omega}$ then there is $P\in\mathcal{A}_{\mathrm{CDZ}}(M)$ such that $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} P : A$. The proof is by induction on P.

Case $P \equiv \bot$. Trivial.

Case $P \equiv \lambda x.P'$. Then $M \twoheadrightarrow_\beta \lambda x.M'$ and $P'\in\mathcal{A}_{\mathrm{CDZ}}(M')$. From $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} P : A$ we get $\Gamma, z{:}\omega, x{:}B_i \vdash^{\mathrm{CDZ}} P' : C_i$ and $\bigcap_{i\in I}(B_i \to C_i) \leq_{\mathrm{CDZ}} A$ for some I and $B_i, C_i\in\mathbb{T}^{\mathrm{CDZ}}$ by Theorem 15.1.2(iii). By induction for each $i\in I$ there is a $P_i\in\mathcal{A}_{\mathrm{CDZ}}(M'[z := N])$ such that $\Gamma, x{:}B_i \vdash^{\mathrm{CDZ}} P_i : C_i$. Let $P_i = \dagger(M_i)$, where $M'[z := N] \twoheadrightarrow_\beta M_i$ and let M'' be a common reduct of the M_i and $P'' \equiv \dagger(M_i)$. Then $P''\in\mathcal{A}_{\mathrm{CDZ}}(M'[z := N])$ and $\Gamma, x{:}B_i \vdash^{\mathrm{CDZ}} P'' : C_i$, for all $i\in I$, by lemma 16.5.6(i). Clearly $\lambda x.P'' : \in\mathcal{A}_{\mathrm{CDZ}}(M[z := N])$ and by construction $\Gamma \vdash^{\mathrm{CDZ}} \lambda x.P'' : A$.

Case $P \equiv x\vec{P}$ then $M \twoheadrightarrow_\beta x\vec{M}$ and $\vec{P}\in\mathcal{A}_{\mathrm{CDZ}}(\vec{M})$. From $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} P : A$ we get $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} x : \vec{B} \to A$ and $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} \vec{P} : \vec{B}$ by Theorem 15.1.16(?). By induction there are $\vec{P'}\in\mathcal{A}_{\mathrm{CDZ}}(\vec{M}[z := N])$ such that $\Gamma \vdash^{\mathrm{CDZ}} \vec{P'} : \vec{B}$. If $x \neq z$ we are done since $x\vec{P'}\in\mathcal{A}_{\mathrm{CDZ}}(M[z := N])$ and we can derive $\Gamma \vdash^{\mathrm{CDZ}} x\vec{P'} : A$

using ($\to$E). Otherwise $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} z : \vec{B} \to A$ implies $\omega \leq_{\mathrm{CDZ}} \vec{B} \to A$ by Theorem 15.1.16(**??**). Being ∇^{CDZ} β-sound by Theorem 15.1.12(ii) from $\omega \sim_{\mathrm{CDZ}} \vec{\varphi} \to \omega$ we obtain $\vec{B} \leq_{\mathrm{CDZ}} \vec{\varphi}$ and $\omega \leq_{\mathrm{CDZ}} A$ by Lemma 14.1.10(iv). So we get $\Gamma \vdash^{\mathrm{CDZ}} \vec{P'} : \vec{\varphi}$, i.e. $\vec{M}[z := N]{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma}$. By Definition 16.5.10(**??**) $N{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma}$ and $\vec{M}[z := N]{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma}$ imply $M[z := N]{\in}[\omega]^{\mathrm{CDZ}}_{\Gamma}$. Since $\omega \leq_{\mathrm{CDZ}} A$ we get $M[z := N]{\in}[A]^{\mathrm{CDZ}}_{\Gamma}$.

The proof for $\triangledown = \mathrm{DHM}$ is similar but easier than that for $\triangledown = \mathrm{CDZ}$. In the case $P \equiv z\vec{P}$ it follows from Definition 16.5.10(**??**) that $N{\in}[\![\omega]\!]^{\mathrm{DHM}}_{\Gamma}$ implies $M[z := N]{\in}[A]^{\mathrm{DHM}}_{\Gamma}$.

(ii) We treat the cases related to $A{\to}B{\leq}\Omega{\to}\Omega$ in AO, $(\omega{\to}\varphi){\sim}\varphi, \omega{\sim}(\varphi{\to}\omega)$ in CDZ, $(\varphi{\to}\varphi) \cap (\omega{\to}\omega){\sim}\varphi$ in HR, and $(\omega{\to}\omega){\sim}\omega$ in Park.

Proof of $[\![A{\to}B]\!]^{\mathrm{AO}}_{\Gamma} \subseteq [\![\Omega{\to}\Omega]\!]^{\mathrm{AO}}_{\Gamma}$. If $B \sim_{\mathrm{AO}} \Omega$, then

$$[\![A{\to}B]\!]^{\mathrm{AO}}_{\Gamma} = [A{\to}B]^{\mathrm{AO}}_{\Gamma} \subseteq [\Omega{\to}\Omega]^{\mathrm{AO}}_{\Gamma} = [\![\Omega{\to}\Omega]\!]^{\mathrm{AO}}_{\Gamma}.$$

If on the other hand $B \not\sim_{\mathrm{AO}} \Omega$, then $M{\in}[\![A{\to}B]\!]^{\mathrm{AO}}_{\Gamma}$. Write $\Gamma' = \Gamma, z{:}A$. Then $z{\in}[A]^{\mathrm{AO}}_{\Gamma'}$, hence $z{\in}[\![A]\!]^{\mathrm{AO}}_{\Gamma'}$. So $Mz{\in}[\![B]\!]^{\mathrm{AO}}_{\Gamma'} \subseteq [B]^{\mathrm{AO}}_{\Gamma'}$ and therefore by Lemma 16.5.9 we have $M{\in}[A{\to}B]^{\mathrm{AO}}_{\Gamma} \subseteq [\Omega{\to}\Omega]^{\mathrm{AO}}_{\Gamma} = [\![\Omega{\to}\Omega]\!]^{\mathrm{AO}}_{\Gamma}$.

Proof of $[\![\omega{\to}\varphi]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma}$. We have

$$\begin{array}{rcll} [\![\omega{\to}\varphi]\!]^{\mathrm{CDZ}}_{\Gamma} & \subseteq & [\omega{\to}\varphi]^{\mathrm{CDZ}}_{\Gamma}, & \text{by Lemma 16.5.11(ii)}, \\ & = & [\varphi]^{\mathrm{CDZ}}_{\Gamma}, & \text{since } \omega{\to}\varphi \sim_{\mathrm{CDZ}} \varphi, \\ & = & [\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma}, & \text{by Definition 16.5.10.} \end{array}$$

Proof of $[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\![\omega{\to}\varphi]\!]^{\mathrm{CDZ}}_{\Gamma}$. Suppose $\Gamma' \supseteq \Gamma$ and $N{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'}$ in order to show $MN{\in}[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}$. By Definition 16.5.10 $[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma} = [\varphi]^{\mathrm{CDZ}}_{\Gamma}$. If $M{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma}$ then there is $P{\in}\mathcal{A}_{\mathrm{CDZ}}(M)$ such that $\Gamma \vdash^{\mathrm{CDZ}} P : \varphi$. We will show $MN{\in}[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}$ by cases of P

Case $P \equiv \bot$. By Proposition 16.5.5(i) one has $\varphi \sim_{\mathrm{CDZ}} \Omega$. Then $[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'} = [\varphi]^{\mathrm{CDZ}}_{\Gamma'} = \Lambda$. Hence $MN{\in}[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}$.

Case $P \equiv \lambda z.P'$. Then $M \twoheadrightarrow_{\beta} \lambda z.M'$ and $P'{\in}\mathcal{A}_{\mathrm{CDZ}}(M')$. From $\Gamma \vdash^{\mathrm{CDZ}} P : \varphi$ we get $\Gamma, z{:}\omega \vdash^{\mathrm{CDZ}} P' : \varphi$ by Theorem 15.1.16(iii), since $\varphi \sim_{\mathrm{CDZ}} \omega{\to}\varphi$. This implies $M'{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma,z:\omega}$. We may assume that $z \notin \mathrm{dom}(\Gamma')$. Then also $M'{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma',z:\omega}$. Therefore

$$\begin{array}{rcll} MN & \twoheadrightarrow_{\beta} & (\lambda z.M')N & \\ & \to_{\beta} & M'[z := N] & \\ & \in & [\varphi]^{\mathrm{CDZ}}_{\Gamma'}, & \text{by (i)}, \\ & = & [\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}. & \end{array}$$

Case $P \equiv x\vec{P}$. Notice that $\Gamma \vdash^{\mathrm{CDZ}} P : \varphi$ implies $\Gamma \vdash^{\mathrm{CDZ}} P : \omega{\to}\varphi$, since $\varphi \sim_{\mathrm{CDZ}} \omega{\to}\varphi$. Take arbitrary $\Gamma' \supseteq \Gamma$ and $N{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'}$. By Lemma 16.5.11(ii) $[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'} \subseteq [\omega]^{\mathrm{CDZ}}_{\Gamma'}$, hence there is $P'{\in}\mathcal{A}_{\mathrm{CDZ}}(N)$ such that $\Gamma' \vdash^{\mathrm{CDZ}} P' : \omega$. We get $\Gamma' \vdash^{\mathrm{CDZ}} PP' : \varphi$. As $PP'{\in}\mathcal{A}_{\mathrm{CDZ}}(MN)$ we conclude that $MN{\in}[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}$.

Proof of $[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma}$. We have $[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\varphi{\to}\omega]^{\mathrm{CDZ}}_{\Gamma}$ by Lemma 16.5.11(ii) and $[\varphi{\to}\omega]^{\mathrm{CDZ}}_{\Gamma} = [\omega]^{\mathrm{CDZ}}_{\Gamma}$, as $\varphi{\to}\omega \sim_{\mathrm{CDZ}} \omega$, using Definition 16.5.7. Moreover using Definition 16.5.10 it follows that

$$\begin{array}{rcl} [\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} & = & \{M \mid \forall\Gamma' \Supseteq \Gamma, \forall N{\in}[\![\varphi]\!]^{\mathrm{CDZ}}_{\Gamma'}.MN{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'}\} \\ & = & \{M \mid \forall\Gamma' \Supseteq \Gamma, \forall N{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.MN{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'}\} \\ & \subseteq & \{M \mid \forall\Gamma' \Supseteq \Gamma, \forall N,\vec{N}{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.MN\vec{N}{\in}[\omega]^{\mathrm{CDZ}}_{\Gamma'}\} \end{array}$$

From $[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\omega]^{\mathrm{CDZ}}_{\Gamma}$ and

$$[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq \{M \mid \forall\Gamma' \Supseteq \Gamma, \forall N,\vec{N}{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.MN\vec{N}{\in}[\omega]^{\mathrm{CDZ}}_{\Gamma'}\}$$

we can conclude

$$[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq \{M \mid \forall\Gamma' \Supseteq \Gamma, \vec{N}{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.M\vec{N}{\in}[\omega]^{\mathrm{CDZ}}_{\Gamma'}\} = [\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma}.$$

Proof of $[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma} \subseteq [\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma}$. Again using Definition 16.5.10 it one has

$$\begin{array}{rcl} M{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma} & \Rightarrow & \forall\Gamma' \Supseteq \Gamma, \forall N,\vec{N}{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.MN\vec{N}{\in}[\omega]^{\mathrm{CDZ}}_{\Gamma'} \\ & \Rightarrow & \forall\Gamma' \Supseteq \Gamma, \forall N{\in}[\varphi]^{\mathrm{CDZ}}_{\Gamma'}.MN{\in}[\![\omega]\!]^{\mathrm{CDZ}}_{\Gamma'} \\ & \Rightarrow & M{\in}[\![\varphi{\to}\omega]\!]^{\mathrm{CDZ}}_{\Gamma}. \end{array}$$

Proof of $[\![(\varphi{\to}\varphi)\cap(\omega{\to}\omega)]\!]^{\mathrm{HR}}_{\Gamma} \subseteq [\![\varphi]\!]^{\mathrm{HR}}_{\Gamma}$. By Lemma 16.5.11(ii) one has

$$\begin{array}{rcl} [\![(\varphi{\to}\varphi)\cap(\omega{\to}\omega)]\!]^{\mathrm{HR}}_{\Gamma} & \subseteq & [(\varphi{\to}\varphi)\cap(\omega{\to}\omega)]^{\mathrm{HR}}_{\Gamma} \\ & = & [\varphi]^{\mathrm{HR}}_{\Gamma} \\ & = & [\![\varphi]\!]^{\mathrm{HR}}_{\Gamma} \end{array}$$

by Definition 16.5.7, $(\varphi{\to}\varphi)\cap(\omega{\to}\omega) \sim_{\mathrm{HR}} \varphi$ and Definition 16.5.10.

Proof of $[\![\varphi]\!]^{\mathrm{HR}}_{\Gamma} \subseteq [\![(\varphi{\to}\varphi)\cap(\omega{\to}\omega)]\!]^{\mathrm{HR}}_{\Gamma}$. Let $\Gamma' \Supseteq \Gamma$.

$$\begin{array}{rcl} M{\in}[\![\varphi]\!]^{\mathrm{HR}}_{\Gamma} & \Rightarrow & M{\in}[\varphi]^{\mathrm{HR}}_{\Gamma} \\ & \Rightarrow & \exists P{\in}\mathcal{A}_{\mathrm{HR}}(M)\ \Gamma \vdash^{\mathrm{HR}} P : \varphi\ (*), \quad \text{by Definition 16.5.7.} \end{array}$$

$$\begin{array}{rcl} N{\in}[\![\varphi]\!]^{\mathrm{HR}}_{\Gamma'} & \Rightarrow & N{\in}[\varphi]^{\mathrm{HR}}_{\Gamma'} \\ & \Rightarrow & \exists P'{\in}\mathcal{A}_{\mathrm{HR}}(N)\ \Gamma' \vdash^{\mathrm{HR}} P' : \varphi\ (**), \quad \text{by Definition 16.5.7.} \end{array}$$

Let $\hat{P} \equiv \Phi PP'$ if P is a λ-abstraction and $\hat{P} \equiv PP'$ otherwise.

$$\begin{array}{rcll} (*) \text{ and } (**) & \Rightarrow & \Gamma' \vdash^{\mathrm{HR}} \hat{P} : \varphi, & \text{by (Ax-}\Phi\text{-HR)}, (\leq_{\nabla\ \mathrm{HR}}), (\to\mathrm{E}) \\ & \Rightarrow & MN{\in}[\varphi]^{\mathrm{HR}}_{\Gamma'}, & \text{since } \hat{P}{\in}\mathcal{A}_{\mathrm{HR}}(MN) \\ & \Rightarrow & MN{\in}[\![\varphi]\!]^{\mathrm{HR}}_{\Gamma'} & \\ & \Rightarrow & M{\in}[\![\varphi \to \varphi]\!]^{\mathrm{HR}}_{\Gamma} & \end{array}$$

$$\begin{array}{rcl} N\in[\![\omega]\!]^{\mathrm{HR}}_{\Gamma'} & \Rightarrow & N\in[\omega]^{\mathrm{HR}}_{\Gamma'} \\ & \Rightarrow & \exists P'\in\mathcal{A}_{\mathrm{HR}}(N)\ \Gamma'\vdash^{\mathrm{HR}} P':\omega\ (***),\quad \text{by Definition 16.5.7.} \\ \vec{N}\in[\![\varphi]\!]^{\mathrm{HR}}_{\Gamma'} & \Rightarrow & \vec{N}\in[\varphi]^{\mathrm{HR}}_{\Gamma'} \\ & \Rightarrow & \exists\vec{P}\in\mathcal{A}_{\mathrm{HR}}(\vec{N})\ \Gamma'\vdash^{\mathrm{HR}}\vec{P}:\vec{\varphi}\ (****),\quad \text{by Definition 16.5.7.} \end{array}$$

Let $\hat{P}\equiv\Phi PP'\vec{P}$ if P is a λ-abstraction and $\hat{P}\equiv PP'\vec{P}$ otherwise.

$$\begin{array}{rcll} (*),(***)\text{ and }(****) & \Rightarrow & \Gamma'\vdash^{\mathrm{HR}}\hat{P}:\omega, & \text{by (Ax-}\Phi\text{-HR), }(\leq_{\nabla\ \mathrm{HR}}),\ (\to\mathrm{E}) \\ & \Rightarrow & MN\vec{N}\in[\omega]^{\mathrm{HR}}_{\Gamma'} & \text{since } \hat{P}\in\mathcal{A}_{\mathrm{HR}}(MN\vec{N}) \\ & \Rightarrow & MN\in[\![\omega]\!]^{\mathrm{HR}}_{\Gamma'} & \\ & \Rightarrow & M\in[\![\omega\to\omega]\!]^{\mathrm{HR}}_{\Gamma}. & \end{array}$$

Proof of $[\![\omega\to\omega]\!]^{\mathrm{Park}}_{\Gamma}\subseteq[\![\omega]\!]^{\mathrm{Park}}_{\Gamma}$. Let $M\in[\![\omega\to\omega]\!]^{\mathrm{Park}}_{\Gamma}$ and $\Gamma'=\Gamma,z:\omega$, where $z\notin\mathrm{FV}(M)$.

$$\begin{array}{rcl} z\in[\omega]^{\mathrm{Park}}_{\{z:\omega\}} & \Rightarrow & z\in[\![\omega]\!]^{\mathrm{Park}}_{\{z:\omega\}} \\ & \Rightarrow & Mz\in[\![\omega]\!]^{\mathrm{Park}}_{\Gamma'} \\ & \Rightarrow & Mz\in[\omega]^{\mathrm{Park}}_{\Gamma'} \\ & \Rightarrow & M\in[\omega]^{\mathrm{Park}}_{\Gamma},\quad \text{by Lemma 16.5.9 and } (\omega\to\omega)\leq_{\mathrm{Park}}\omega, \\ & \Rightarrow & M\in[\![\omega]\!]^{\mathrm{Park}}_{\Gamma}. \end{array}$$

Proof of $[\![\omega]\!]^{\mathrm{Park}}_{\Gamma}\subseteq[\![\omega\to\omega]\!]^{\mathrm{Park}}_{\Gamma}$. Let $\Gamma'\supseteq\Gamma$. Then we have

$$\begin{array}{rcl} M\in[\![\omega]\!]^{\mathrm{Park}}_{\Gamma} & \Rightarrow & M\in[\omega]^{\mathrm{Park}}_{\Gamma} \\ & \Rightarrow & \exists P\in\mathcal{A}_{\mathrm{Park}}(M)\ \Gamma\vdash^{\mathrm{Park}} P:\omega\ (*),\quad \text{by Definition 16.5.7.} \end{array}$$

$$\begin{array}{rcl} N\in[\![\omega]\!]^{\mathrm{Park}}_{\Gamma'} & \Rightarrow & N\in[\omega]^{\mathrm{Park}}_{\Gamma'} \\ & \Rightarrow & \exists P'\in\mathcal{A}_{\mathrm{Park}}(N)\ \Gamma'\vdash^{\mathrm{Park}} P':\omega\ (**),\quad \text{by Definition 16.5.7.} \end{array}$$

Let $\hat{P}\equiv\Phi PP'$ if P is a λ-abstraction and $\hat{P}\equiv PP'$ otherwise.

$$\begin{array}{rcll} (*)\text{ and }(**) & \Rightarrow & \Gamma'\vdash^{\mathrm{Park}} P:\omega, & \text{by (Ax-}\Phi\text{), }(\leq_{\mathrm{Park}}),\ (\to\mathrm{E}) \\ & \Rightarrow & MN\in[\omega]^{\mathrm{Park}}_{\Gamma'}, & \text{since } \hat{P}\in\mathcal{A}_{\mathrm{Park}}(MN) \\ & \Rightarrow & MN\in[\![\omega]\!]^{\mathrm{Park}}_{\Gamma'} & \\ & \Rightarrow & M\in[\![\omega\to\omega]\!]^{\mathrm{Park}}_{\Gamma}.\ \blacksquare & \end{array}$$

16.5.13. Definition (Semantic Satisfiability). Let ρ be a mapping from term variables to terms and write $[\![M]\!]_\rho=M[\vec{x}:=\rho(\vec{x})]$, where $\vec{x}=\mathrm{FV}(M)$.

(i) $\nabla,\rho,\Gamma\models M:A$ iff $[\![M]\!]_\rho\in[\![A]\!]^{\nabla}_{\Gamma}$;

(ii) $\nabla,\rho,\Gamma'\models\Gamma$ iff $\nabla,\rho,\Gamma'\models x:B$ for all $(x{:}B)\in\Gamma$;

(iii) $\Gamma\models^{\nabla} M:A$ iff $\nabla,\rho,\Gamma'\models\Gamma$ implies $\nabla,\rho,\Gamma'\models M:A$ for all mappings ρ and all ∇-bases Γ'.

In line with the previous remarks, the following result can be constructed also as the soundness of the natural semantics of intersection types over a particular Kripke applicative structure, where bases play the role of worlds.

16.5.14. LEMMA (Soundness). *Let* $\bigtriangledown\in\{BCD, Scott, CDZ, DHM, AO, Park, HR\}$ *then* $\Gamma \vdash^{\nabla} M : A$ *implies* $\Gamma \models^{\nabla} M : A$.

PROOF. The proof is by induction on the derivation of $\Gamma \vdash^{\nabla} M : A$.
Cases (Ax), and (Ax-Ω) are immediate.
Cases ($\to$E) and ($\cap$I) follow by induction.
Case ($\leq$) is ok by Lemma 16.5.12(ii).
If we are in case ($\to$I), suppose that $M \equiv \lambda y.R$, $A \equiv B{\to}C$ and $\Gamma, y : B \vdash^{\nabla} R : C$ has been derived.
Let $\bigtriangledown \neq$ AO or $C\not\sim_{\mathrm{AO}}\Omega$. Let $\bigtriangledown, \rho, \Gamma' \models \Gamma$. We must show $[\![\lambda y.R]\!]_\rho\in[\![B{\to}C]\!]^{\bigtriangledown}_{\Gamma'}$.Let $\Gamma''\supseteq \Gamma'$ and $T\in[\![B]\!]^{\bigtriangledown}_{\Gamma''}$. Then by the induction hypothesis $[\![R]\!]_{\rho[y:=T]}\in[\![C]\!]^{\bigtriangledown}_{\Gamma''}$. We may assume $y\notin\rho(x)$ for all $x\in\mathrm{dom}(\Gamma)$. Then $[\![\lambda y.R]\!]_\rho T \to_\beta [\![R]\!]_{\rho[y:=T]}$ and hence $[\![\lambda y.R]\!]_\rho T\in[\![C]\!]^{\bigtriangledown}_{\Gamma''}$. So $[\![\lambda y.R]\!]_\rho\in[\![B{\to}C]\!]^{\bigtriangledown}_{\Gamma'}$.
The case $\bigtriangledown =$ AO and $C \sim_{\mathrm{AO}} \Omega$ follows easily observing that $\lambda x.\bot\in\mathcal{A}_{\mathrm{AO}}([\![\lambda y.R]\!]_\rho)$ for all ρ and we can derive $\vdash^{\mathrm{AO}} \lambda x.\bot : B{\to}C$ using (Ax-Ω), ($\to$I) and ($\leq_{\mathrm{AO}}$). We conclude $[\![M]\!]_\rho\in[A]^{\bigtriangledown}_{\Gamma}$ which implies $[\![M]\!]_\rho\in[\![A]\!]^{\bigtriangledown}_{\Gamma}$ by Definition 16.5.10(**??**). ∎

Now we can prove the converse of proposition 16.5.8.

16.5.15. PROPOSITION. *Let* $\bigtriangledown\in\{BCD, Scott, CDZ, DHM, AO, Park, HR\}$. *Then*

$$\Gamma \vdash^{\nabla} M : A \Rightarrow M\in[A]^{\bigtriangledown}_{\Gamma}.$$

PROOF. Let $\rho_0(x) = x$. By Lemma 16.5.11(i) $\bigtriangledown, \rho_0, \Gamma \models \Gamma$. Then $\Gamma \vdash^{\nabla} M : A$ implies $M = [\![M]\!]_{\rho_0}\in[\![A]\!]^{\bigtriangledown}_{\Gamma}$ by Lemma 16.5.14. So we conclude $M\in[A]^{\bigtriangledown}_{\Gamma}$ by Lemma 16.5.11(ii). ∎

16.5.16. THEOREM (Approximation Theorem). *Let*

$$\bigtriangledown\in\{BCD, Scott, CDZ, DHM, AO, Park, HR\}.$$

Then $\Gamma \vdash^{\nabla} M : A \;\Leftrightarrow\; \exists P\in\mathcal{A}_{\bigtriangledown}(M).\Gamma \vdash^{\bigtriangledown} P : A.$
PROOF. By Propositions 16.5.8 and 16.5.15. ∎

16.5.17. COROLLARY. *Let* $M\in\Lambda$ *be untyped and* $\Gamma_\rho = \{x{:}B \mid B\in\rho(x)\}$. *Let*

$$\bigtriangledown\in\{BCD, Scott, CDZ, DHM, AO, Park, HR\}.$$

Then $[\![M]\!]^{\bigtriangledown}_{\rho} = \{A\in\mathcal{T}^{\bigtriangledown} \mid \exists P\in\mathcal{A}_{\bigtriangledown}(M).\Gamma_\rho \vdash^{\bigtriangledown} P : A\}$.
PROOF. By theorem 16.4.4, Exercise 16.6.4 (for $\nabla =$ AO) and the Approximation Theorem. ∎

Theorem 16.5.16 was first proved for $\bigtriangledown =$ BCD in Barendregt et al. [1983], for $\bigtriangledown =$ Scott in Ronchi Della Rocca [1988], for $\bigtriangledown =$ CDZ in Coppo et al. [1987], for $\bigtriangledown =$ AO in Abramsky and Ong [1993], for $\bigtriangledown =$ Park and $\bigtriangledown =$ HR in Honsell and Ronchi Della Rocca [1992].

Some applications of the Approximation Theorems

As discussed in Section 16.4 type theories give rise in a natural way to *filter λ-models*. Properties of $\mathcal{F}^\nabla$ with $\nabla\in\{\mathrm{BCD}, \mathrm{Scott}, \mathrm{CDZ}, \mathrm{AO}, \mathrm{Park}, \mathrm{HR}\}$ can be easily derived using Theorems 16.5.8 and 16.5.15. For instance, one can check the following.

- The models $\mathcal{F}^{\mathrm{BCD}}, \mathcal{F}^{\mathrm{Scott}}, \mathcal{F}^{\mathrm{CDZ}}, \mathcal{F}^{\mathrm{DHM}}$ are sensible (see Barendregt [1984]).
- The top element in $\mathcal{F}^{\mathrm{AO}}$ is the interpretation of the terms of order ∞.
- The model $\mathcal{F}^{\mathrm{Park}}$ characterizes the terms reducible to closed terms.
- The model $\mathcal{F}^{\mathrm{HR}}$ characterizes the terms reducible to λI-terms.

The rest of this section is devoted to the proof of these properties. Other uses of the Approximation Theorem can be found in the corresponding relevant papers, i.e. Barendregt et al. [1983], Coppo et al. [1987], Ronchi Della Rocca [1988], Abramsky and Ong [1993], Honsell and Ronchi Della Rocca [1992].

16.5.18. Theorem. *The models $\mathcal{F}^\nabla$ with $\nabla\in\{BCD, Scott, CDZ, DHM\}$ are sensible, i.e. for all unsolvable terms M, N one has*

$$[\![M]\!]_\rho^{\mathcal{F}^\nabla} = [\![N]\!]_\rho^{\mathcal{F}^\nabla}.$$

Proof. It follows immediately from Corollary 16.5.17 of the Approximation Theorem, the fact that $\perp$ is the only approximant of an unsolvable term for the mapping α, and Proposition 16.5.5(i). ∎

Let us recall the definition of term of order ∞, see Longo [1983].

16.5.19. Definition. An untyped lambda term M is *of order* ∞ iff

$$\forall n \exists M'. M \twoheadrightarrow_\beta \lambda x_1 \ldots \lambda x_n . M'.$$

16.5.20. Theorem. *Let M be an untyped lambda term. Then the following are equivalent.*

(i) *M is of order ∞.*
(ii) *$\vdash^{AO} M : A$ for all types $A\in\mathbb{T}^{AO}$.*
(iii) *$[\![M]\!]_\rho^{AO} = \Omega$ for all valuations ρ.*

Proof. ((i)$\Leftrightarrow$(ii)). It is easy to check by structural induction on types that

$$\forall A\in\mathbb{T}^{\mathrm{AO}} \exists n\in\mathbb{N}. (\Omega^n \to \Omega) \leq_{\mathrm{AO}} A.$$

So by the Approximation Theorem it suffices to show that if $P\in\Lambda\perp$ is an approximate normal form we have

$$\vdash^{\mathrm{AO}} P : (\Omega^n \to \Omega) \;\Leftrightarrow\; P \equiv \lambda x_1 \ldots \lambda x_n . P' \text{ for some } P'.$$

($\Leftarrow$) By axiom (Ω) and rule ($\to$I). ($\Rightarrow$) Assume towards a contradiction that $P \equiv \lambda x_1 \dots \lambda x_m.P'$ for $m < n$ and P' is of the form $\perp$ or $x\vec{P}$. Then by Theorem 15.1.16(iii)

$$\vdash^{\mathrm{AO}} P : (\Omega^n \to \Omega) \;\Rightarrow\; \{x_1{:}\Omega, \dots, x_m{:}\Omega\} \vdash^{\mathrm{AO}} P' : (\Omega^{n-m} \to \Omega).$$

But this latter judgment can neither be derived if $P' \equiv \perp$, by Propositions 16.5.5(i) and 14.1.10(ii), nor if $P' \equiv x\vec{P}$, by Theorem 15.1.16(i) and (ii).

((ii)$\Leftrightarrow$(iii)).[Proof not yet finished.] ∎

16.5.21. THEOREM. *A term M reduces to a closed term iff $\vdash^{Park} M : \omega$.*

PROOF. By the Approximation Theorem it suffices to check that if $P{\in}\Lambda\Phi$ and $\mathcal{V}$ is a set of term variables:

$$\{x{:}\omega \mid x{\in}\mathcal{V}\} \vdash^{\mathrm{Park}} P : \omega \text{ iff } FV(P) \subseteq \mathcal{V}.$$

The proof is by structural induction on P. Let $\Gamma = \{x{:}\omega \mid x{\in}\mathcal{V}\}$. The key property is that $\omega \sim_{\mathrm{Park}} \omega \to \omega$.
The case $P \equiv \lambda y.P'$ follows by Theorem 15.1.16(**??**) and using the induction hypothesis.
If $P \equiv y\vec{P}$ then by Theorem 15.1.16(**??**) and (**??**) we need $y{\in}\mathcal{V}$. Since Park is a beta theory we get as the proof of Lemma 16.5.6(i) $\Gamma \vdash^{\mathrm{Park}} \vec{P} : \vec{\omega}$, so we conclude using the induction hypothesis.
Similarly if $P \equiv \Phi P'\vec{P}$ then by Theorem 15.1.16(**??**) and Proposition 16.5.5(ii) we have $\Gamma \vdash^{\mathrm{Park}} P' : \omega$ and $\Gamma \vdash^{\mathrm{Park}} \vec{P} : \vec{\omega}$. ∎

Lastly we work out the characterization of terms reducible to λI-terms.
We define the set of terms we want to characterize and the set obtained by adding the constant Φ to it.

16.5.22. DEFINITION. (i) The set $\Lambda\mathsf{I}\beta$ of $\lambda\mathsf{I}\beta$-terms is the subset of Λ such that $M{\in}\Lambda\mathsf{I}\beta$ iff M reduces to a term in which all abstracted variables occur at least once (i.e. M reduces to a λI-term).

(ii) The set $\Lambda\mathsf{I}\beta\Phi$ of $\lambda\mathsf{I}\beta\Phi$-terms is obtained by adding the constant Φ to the formation rules of $\lambda\mathsf{I}\beta$-terms. ∎

In the following key lemma we show that each approximate normal form $P{\in}\Lambda\mathsf{I}\beta\Phi$ is typable with φ from the HR-basis all whose predicates are φ. To properly deal with the structural induction on approximate normal forms (in the case of abstractions) we also show that if $x{\in}P$ then P is typable with ω from the HR-basis containing $x{:}\omega$ and all whose other predicates are φ. This is useful since $\varphi \sim_{\mathrm{HR}} (\varphi{\to}\varphi) \cap (\omega{\to}\omega)$.

16.5.23. LEMMA. *Let $P{\in}\Lambda\Phi$ be an anf, and $\Gamma_\varphi = \{y{:}\varphi \mid y{\in}\mathrm{Var}\}$. Then*

(i) $P{\in}\Lambda\mathsf{I}\beta\Phi \;\Leftrightarrow\; \Gamma_\varphi \vdash^{HR} P : \varphi$.

(ii) *Let $P{\in}\Lambda\mathsf{I}\beta\Phi$. Then $x{\in}P \;\Leftrightarrow\; \Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{HR} P : \omega$.*

PROOF. (i) and (ii) can be simultaneously proved by induction on P.

Case $P \equiv y$ is trivial.

Case $P \equiv \lambda z.P'$. For ($\Rightarrow$) notice that $P{\in}\Lambda|\beta\Phi$ implies $P'{\in}\Lambda|\beta\Phi$ and $z{\in}P'$. Then we have by induction $\Gamma_\varphi \vdash^{\mathrm{HR}} P' : \varphi$ and $\Gamma_\varphi \uplus \{z{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$, so by rules ($\to$I), ($\cap$I) and ($\leq_{\nabla\ \mathrm{HR}}$) we get $\Gamma_\varphi \vdash^{\mathrm{HR}} P : \varphi$. If $x{\in}P$ then $x{\in}P'$ and by induction $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$, so by rules ($\to$I) and ($\leq_{\nabla\ \mathrm{HR}}$) we get $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P : \omega$. For ($\Leftarrow$) $\Gamma_\varphi \vdash^{\mathrm{HR}} P : \varphi$ implies $\Gamma_\varphi \vdash^{\mathrm{HR}} P' : \varphi$ and $\Gamma_\varphi \uplus \{z{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$ by Theorem 15.1.16(**??**) since $\varphi \sim_{\mathrm{HR}} (\varphi{\to}\varphi) \cap (\omega{\to}\omega)$ and ∇^{HR} is β-sound [and beta?]. By induction $P'{\in}\Lambda|\beta\Phi$ and $z{\in}P'$ so we get $P{\in}\Lambda|\beta\Phi$. Moreover $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P : \omega$ implies $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$ so we get by induction $x{\in}P'$, i.e. $x{\in}P$.

Case $P \equiv \chi P'\vec{P}$ where $\chi{\in}\{\Phi, z\}$. For ($\Rightarrow$) notice that $P{\in}\Lambda|\beta\Phi$ implies $P'{\in}\Lambda|\beta\Phi$ and $\vec{P}{\in}\Lambda|\beta\Phi$. Then we have by induction $\Gamma_\varphi \vdash^{\mathrm{HR}} P' : \varphi$ and $\Gamma_\varphi \vdash^{\mathrm{HR}} P_i : \varphi$ for all i, so by axiom either (Ax) or (Ax-Φ-HR) and rules ($\to$E), ($\leq_{\nabla\ \mathrm{HR}}$) we get $\Gamma_\varphi \vdash^{\mathrm{HR}} P : \varphi$. If $x{\in}P$ then either $\chi \equiv x$ or $x{\in}P'\vec{P}$. In the first case $x{:}\omega \vdash^{\mathrm{HR}} x : \omega$ so by rules (*weakening*), ($\to$E), ($\leq_{\nabla\ \mathrm{HR}}$) we get $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P : \omega$. In the second case by induction either $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$ or $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P_i : \omega$ for some i, so by axiom either (Ax) or (Ax-Φ-HR) and rules ($\to$E), ($\leq_{\nabla\ \mathrm{HR}}$) we get $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P : \omega$. For ($\Leftarrow$) notice that $\Gamma_\varphi \vdash^{\mathrm{HR}} \chi : A$ implies $\varphi \leq_{\mathrm{HR}} A$ by Theorem 15.1.16(**??**) when $\chi \equiv z$ or by Proposition 16.5.5(i) when $\chi \equiv \Phi$. So $\Gamma_\varphi \vdash^{\mathrm{HR}} P : \varphi$ implies $\Gamma_\varphi \vdash^{\mathrm{HR}} P' : \varphi$ and $\Gamma_\varphi \vdash^{\mathrm{HR}} P_i : \varphi$ for all i by Theorem 15.1.16(**??**) since $\varphi \sim_{\mathrm{HR}} (\varphi{\to}\varphi) \cap (\omega{\to}\omega)$ and ∇^{HR} validates CDV. By induction $P'{\in}\Lambda|\beta\Phi$ and $P_i{\in}\Lambda|\beta\Phi$ for all i, so we get $P{\in}\Lambda|\beta\Phi$. By a similar argument $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P : \omega$ and $\chi \not\equiv x$ imply either $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P' : \omega$ or $\Gamma_\varphi \uplus \{x{:}\omega\} \vdash^{\mathrm{HR}} P_i : \omega$ for some i, so we get by induction either $x{\in}P'$ or $x{\in}P_i$ for some i, i.e. $x{\in}P$. ∎

Now we can prove the final result.

16.5.24. THEOREM. *A term $M{\in}\Lambda|\beta$ iff M is typable with φ in the system $\lambda\cap_\Omega^{HR}$ from the HR-basis all whose predicates are φ.*

PROOF. Easy from Theorems 16.5.8, 16.5.15 and Lemma 16.5.23(i). ∎

Theorem 16.5.24 was first proved in Honsell and Ronchi Della Rocca [1992] by purely semantic means.

16.6. Exercises 26.2.2004:1018

16.6.1. A *weak natural type structure* (**WNTS**) is an intersection type structure with element Ω satisfying $(\Omega), (\eta), ({\to}\text{-}\cap)$ and (Ω-lazy). Cf. the definition of **NTS** in 16.2.1. The category **WNTS** has as objects weak natural type structures and as morphisms approximable mappings satisfying condition (†), as for **NTS**. Clearly $\mathcal{T}^{\mathrm{AO}}{\in}\mathbf{WNTS}$.

(i) Let S be an ITS. Prove (cf. Proposition 16.2.3)

$$S{\in}\mathbf{WNTS} \;\Leftrightarrow\; \begin{array}{l}\text{for each } r{\in}S \text{ the relation } \mu_r \subseteq S \times S\\ \text{defined by} \quad s\mu_r t \;\Leftrightarrow\; r \leq s{\to}t\\ \text{is empty or an approximable mapping.}\end{array}$$

(ii) Let S be an ITS. Prove

$$S \in \mathbf{WNTS} \Leftrightarrow \quad \begin{array}{l} \text{for each } r = \bigcap_{i\in I}(A_i \to B_i), I \neq \emptyset \\ \text{the relation } \mu_r \subseteq S \times S \text{ defined by} \\ \qquad s\mu_r t \Leftrightarrow r \leq s \to t \\ \text{is an approaximable mapping.} \end{array}$$

(iii) Similarly to Definition 16.2.12 (i) we define

$$\begin{array}{rcl} F^{\mathcal{S}}(X)(Y) & = & \uparrow\{b\in\mathcal{S} \mid \exists a\in Y.a \to b\in X\} \\ G^{\mathcal{S}}(f) & = & \uparrow\{a\to b\in\mathcal{S} \mid b\in f(\uparrow a)\} \end{array}$$

(In Definition 16.2.12(i) the $\uparrow$ in the first line was missing.) Prove

$$F^{\mathcal{S}}(X)(Y) = \begin{cases} \{b\in\mathcal{S} \mid \exists a\in Y.a\to b\in X\} & \text{if } \Omega\to\Omega\in X \\ \uparrow\Omega & \text{else.} \end{cases}$$

(iv) Prove

$$F^{\mathcal{S}}(\uparrow s_1)(\uparrow s_2) = \begin{cases} \{t\in\mathcal{S} \mid s_1 \leq s_2\to t\} & \text{if } s_1 \leq s_2\to\Omega \\ \uparrow\Omega & \text{else.} \end{cases}$$

Cf. Lemma 16.2.13.

16.6.2. Show that Proposition 16.2.14 remains valid after replacing **NTS** by **WNTS**.

16.6.3. By an example we will show that Proposition 16.2.15 is not valid after replacing **NTS** by **WNTS**.

(i) Let ∇ be the ITT with as constants the set $\{\omega, \Omega\}$ and as special axioms (for all A, B, C, D)

$$\boxed{\begin{array}{rcl} A & \leq & \Omega \\ A\to B & \leq & C\to D \end{array}}$$

Verify that there are 4 equivalence classes modulo $\sim$, with $\leq$ on them given by the diagram of representatives

$$\begin{tikzcd}[column sep=small] & \Omega \arrow[dl, dash] \arrow[dr, dash] & \\ \omega \arrow[dr, dash] & & \Omega\to\Omega \arrow[dl, dash] \\ & (\Omega\to\Omega)\cap\omega & \end{tikzcd}$$

Verify that $\mathcal{T}^{\nabla}\in\mathbf{WNTS}$, but $\mathcal{T}^{\nabla} \notin \mathbf{NTS}$.

(ii) Define $\mu\in\mathbf{MSL}(\mathcal{T}^{\nabla}, \mathcal{T}^{\nabla})$ (referring to the above diagram) by

$$A\,\mu\,B \quad \Leftrightarrow \quad [A \notin \{\omega, \Omega\} \vee B \notin \{\Omega\to\Omega, (\Omega\to\Omega)\cap\omega\}].$$

Note that μ is an extension of $\leq$ such that $\Omega\,\mu\,\omega$. Verify that indeed μ is a **WNTS**-morphism, i.e. μ is an approximable mapping such that (†) of Definition 16.2.1(ii) holds.

(iii) Let $\mathsf{m} = \mathsf{Flt}(\mu)$. Then $\omega \in \mathsf{m}(\uparrow\Omega)$, but $\omega \notin \uparrow\Omega$, hence $\mathsf{m}(\uparrow\Omega) \neq \uparrow\Omega$.
(iv) Prove using exercise 16.6.1(iii) that

$$m(F^{\mathcal{T}^\nabla}(\uparrow\Omega)(\uparrow\Omega)) \neq F^{\mathcal{T}^\nabla}(m(\uparrow\Omega)(m(\uparrow\Omega)).$$

Hence Proposition 16.2.15 fails with **NTS** replaced by **WNTS**.

16.6.4. Prove that Theorem 16.4.4 remains valid if the notion **NTS** is replaced by **WNTS**. [Hint. For the case $M \equiv NL$ distinguish the 2 subcases $\Omega{\to}\Omega \in [\![N]\!]^{\mathcal{S}}_\rho$ and $\Omega{\to}\Omega \notin [\![N]\!]^{\mathcal{S}}_\rho$ and use exercise 16.6.1(iii).]

16.6.5. Let $\nabla^\triangledown$ be an arbitrary type theory such that $\Omega \notin \mathbb{C}^\triangledown$. Show that
$\emptyset \cdot X \ = \ \emptyset$ for all $X \in \mathcal{F}^\triangledown$
$X \cdot \emptyset \ = \ \emptyset$ for all $X \in \mathcal{F}^\triangledown$.

16.6.6. Let $\nabla^\triangledown$ be a theory such that $\nu \in \mathbb{C}^\triangledown$. Show that
$\uparrow \nu \cdot X \ = \ \emptyset$ for all $X \in \mathcal{F}^\triangledown$ iff $\nabla^\triangledown$ is ν-sound.

16.6.7. Let $\nabla^\triangledown$ be a Ω-natural and beta type theory. Show that
$\uparrow \bigcap_{i\in I}(A_i \to B_i) \cdot \uparrow C \ = \ \bigcap_{j\in I} B_j$
where $J = \{i \in I \mid C \leq_\triangledown A_i\}$.

16.6.8. Let $\nabla^\triangledown$ be a theory such that $\nu \in \mathbb{C}^\triangledown$. Show that
$F_{\mathcal{D}}(G_{\mathcal{D}}(\bot \Rightarrow \bot)) = \bot \Rightarrow \bot$ iff $\nabla^\triangledown$ is ν-sound.

16.6.9. Let $\nabla^\triangledown$ be a theory such that $\nu \in \mathbb{C}^\triangledown$ and each type is equivalent to some constant $\psi \in \mathbb{C}^\triangledown$. Show that
$F^\triangledown_{\infty}(\uparrow \nu) = \bigsqcup\{\psi \Rightarrow \psi' \mid \nu \leq_\triangledown \psi{\to}\psi'\}$.

16.6.10. Let $\nabla^\triangledown$ be a theory such that $\Omega \in \mathbb{C}^\triangledown$. Show that
$\uparrow (\Omega \to \Omega) \cdot X = \uparrow \Omega$ for all $X \in \mathcal{F}^\triangledown$ iff
$B \sim_\triangledown \Omega$ whenever $A \to B \sim_\triangledown \Omega \to \Omega$.

16.6.11. Let $\nabla^\triangledown$ be a theory such that $\Omega \in \mathbb{C}^\triangledown$. Show that
$F_{\mathcal{D}}(G_{\mathcal{D}}(\bot \Rightarrow \bot)) = \bot \Rightarrow \bot \ \Rightarrow$
$B \sim_\triangledown \Omega$ whenever $A \to B \sim_\triangledown \Omega \to \Omega$.

16.6.12. Let $\nabla^\triangledown$ be a theory such that $\Omega \in \mathbb{C}^\triangledown$. Show that
$F_{\mathcal{D}}(G_{\mathcal{D}}(\bot \Rightarrow \bot)) = \bot \Rightarrow \bot$ iff
$(\bigcap_{i\in I}(C_i{\to}D_i) \leq_\triangledown A{\to}B \ \&\ \forall i \in I. D_i \sim_\triangledown \Omega) \Rightarrow B \sim_\triangledown \Omega$.

16.6.13. Let $\nabla^\triangledown$ be an arbitrary type theory. Show that
$G^\triangledown(\bigsqcup_{i\in I}(\uparrow A_i \Rightarrow \uparrow B_i)) \ \supseteq \ \uparrow \bigcap_{i\in I}(A_i{\to}B_i)$.

16.6.14. Let $\nabla^\triangledown$ be a Ω-free natural type theory. Show that
$G^\triangledown(\bigsqcup_{i\in I}(\uparrow A_i \Rightarrow \uparrow B_i)) \ = \ \uparrow \bigcap_{i\in I}(A_i{\to}B_i)$.

16.6.15. Let $\nabla^\triangledown$ be a Ω-natural type theory. Show that
$G^\triangledown(\bigsqcup_{i\in I}(\uparrow A_i \Rightarrow \uparrow B_i)) \ = \ \uparrow \bigcap_{i\in I}(A_i{\to}B_i)$.

16.6.16. Let $\nabla^\triangledown$ be a Ω-natural type theory which validates axiom (Ω-η). Show that
$G^\triangledown(\uparrow \Omega \Rightarrow \uparrow \Omega) \ = \ \uparrow \Omega$.

16.6.17. Let $\nabla^\triangledown$ be a Ω-free natural type theory such that $\nu \notin \mathbb{C}^\triangledown$. Show that $F^\triangledown_{\infty}, G^\triangledown$ set up a Galois connection, i.e.:

(i) $F^{\nabla}_{\diamond\diamond} \circ G^{\nabla} \sqsupseteq Id_{[\mathcal{F}^{\nabla} \to_{\perp} \mathcal{F}^{\nabla}]}$;
(ii) $G^{\nabla} \circ F^{\nabla}_{\diamond\diamond} \sqsubseteq Id_{\mathcal{F}^{\nabla}}$.
Moreover show that $F_{\mathcal{D}}$ determines $G_{\mathcal{D}}$ by

$$G_{\mathcal{D}}(f) = \sqcap\{d \mid f \sqsubseteq F_{\mathcal{D}}(d)\}$$

for all strict continuous functions f.

16.6.18. Let ∇^{∇} be a Ω-natural type theory such that $A \to B \sim_{\nabla} \Omega \to \Omega$ implies $B \sim_{\nabla} \Omega$. Show that $F^{\nabla}_{\diamond\diamond}, G^{\nabla}$ set up a Galois connection, i.e.:
(i) $F^{\nabla}_{\diamond\diamond} \circ G^{\nabla} \sqsupseteq Id_{[\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}]}$;
(ii) $G^{\nabla} \circ F^{\nabla}_{\diamond\diamond} \sqsubseteq Id_{\mathcal{F}^{\nabla}}$.
Moreover show that $F_{\mathcal{D}}$ determines $G_{\mathcal{D}}$ by

$$G_{\mathcal{D}}(f) = \sqcap\{d \mid f \sqsubseteq F_{\mathcal{D}}(d)\}$$

for all continuous functions f.

16.6.19. Let $\Omega \in \mathbb{C}^{\nabla}$ and $f \in [\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}]$. Prove that if ∇^{∇} validates BCD, then f is representable iff $F^{\nabla}_{\diamond\diamond}(G^{\nabla}(f)) = f$.

16.6.20. Let $\Omega \notin \mathbb{C}^{\nabla}$ and $f \in [\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}]$. Prove that is ∇^{∇} validates CDV and is ν-sound, then f is representable iff $F^{\nabla}_{\diamond\diamond}(G^{\nabla}(f)) = f$.

16.6.21. Check the following equalities:

$$[\![\lambda x.y]\!]^{\mathrm{CDV}}_{\rho_0} = \emptyset \quad [\![\lambda x.y]\!]^{\mathrm{HL}}_{\rho_1} = \uparrow \omega \quad [\![\lambda x.y]\!]^{\mathrm{EHR}}_{\rho_0} = \uparrow \nu \quad [\![\lambda x.y]\!]^{\mathrm{AO}}_{\rho_0} = \uparrow (\Omega \to \Omega)$$

where $\rho_0(y) = \uparrow \emptyset$ and $\rho_1(y) = \uparrow \omega$.

16.6.22. Prove using the results of Exercise 14.3.17 that $[\![\mathsf{K}^{\infty}]\!]^{\mathrm{AO}}_{\rho} = \mathcal{F}^{\mathrm{AO}}$.

16.6.23. Let TM be the term model of β-equality and $[M]$ the equivalence class of M under β-equality. Let a term M be persistently head normalizing iff $M\vec{N}$ has a head normal form for all terms $\vec{N}$ (see Definition 17.1.2). Prove that the type interpretation domain $\langle TM, \cdot, [\![\ ]\!]^{TM}, TM\rangle$, and the type environment $\mathcal{V}_{\mathrm{Scott}}(\omega) = \{[M] \mid M \text{ is persistently head normalizing}\}$ agree with ∇^{Scott}.

16.6.24. Let TM and $[M]$ be as in Exercise 16.6.23. Prove that the type interpretation domain $\langle TM, \cdot, [\![\ ]\!]^{TM}, TM\rangle$, and the type environment $\mathcal{V}_{\mathrm{Park}}(\omega) = \{[M] \mid M \text{ reduces to a closed term}\}$ agree with ∇^{Park}.

16.6.25. Let TM and $[M]$ be as in Exercise 16.6.23. Let a term M be persistently normalizing iff $M\vec{N}$ has a normal form for all normalizing terms $\vec{N}$ (see Definition 17.1.2). Prove that the type interpretation domain $\langle TM, \cdot, [\![\ ]\!]^{TM}, TM\rangle$, and the type environment $\mathcal{V}_{CDZ}(\omega) = \{[M] \mid M \text{ is persistently normalizing}\}$, $\mathcal{V}_{CDZ}(\varphi) = \{[M] \mid M \text{ is normalizing}\}$ agree with ∇^{CDZ}.

16.6.26. Show by means of examples that the type theories Plotkin e Engeler are not adequate.

16.6.27. Show that $\Sigma(\{\omega\}, \mathrm{CDV})$ is the smallest complete type theory (w.r.t. the order of Figure 14.6).

16.6.28. Show that for all $\triangledown$, x, $A, B \in \mathbb{T}^{\triangledown}$, $\psi \in \mathbb{C}^{\triangledown}$, $A', B' \in \mathbb{T}^{Plo}$:

1. $\models_s^{\triangledown} x : \Omega \to \Omega$;
2. $\not\models_i^{\triangledown} x : \Omega \to \Omega$;
3. $\models_s^{\triangledown} x : \nu$;
4. $\not\models_i^{\triangledown} x : \nu$;
5. $x{:}\psi \cap (A \to B) \models_F^{\triangledown} \lambda y.xy : \psi$;
6. $x{:}\varphi \cap (A' \to B') \not\models_i^{Plo} \lambda y.xy : \varphi$.

16.6.29. Show that if $\nabla^{\triangledown}$ is a Ω-natural type theory which is adequate for the F-semantics, then for all $n \geq 0$ and for all $\psi \in \mathbb{C}^{\triangledown}$ we can find $I \neq \emptyset, A_i^{(1)}, \dots, A_i^{(n)}, B_i$ such that

$$\psi \cap (\Omega^n \to \Omega) \sim_{\triangledown} \bigcap_{i \in I} (A_i^{(1)} \to \dots \to A_i^{(n)} \to B_i).$$

16.6.30. Show that if $\nabla^{\triangledown}$ is a Ω-natural type theory which is adequate for the F-semantics, then the following rule (Dezani-Ciancaglini and Margaria [1986])

$$\text{(Hindley rule)} \quad \frac{\Gamma \vdash^{\triangledown} M : \psi \cap (\Omega^n \to \Omega) \quad x_i \notin \mathrm{FV}(M) \ (1 \leq i \leq n)}{\Gamma \vdash^{\triangledown} \lambda x_1 \dots x_n . M x_1 \dots x_n : \psi}$$

is admissible in $\lambda\cap^{\triangledown}$ for all $\psi \in \mathbb{C}^{\triangledown}$. (Hint: use the result of Exercise 16.6.29).

Chapter 17

Applications 26.2.2004:1018

The type assignment systems introduced in Section 14.2 will be put to use in the present chapter, where we show how to characterize a number of properties of λ-terms using intersection types. In particular, in Sections 17.1 we shall discuss normalization properties. Section 16.5 is devoted to prove a general approximation theorem for intersection type assignment systems. In the Section 17.3 we see how to solve domains equations by building suitable filter structures.

17.1. Characterizing syntactic properties

In this section we will see the intersection type systems at work in the characterization of properties of λ-terms. Since types are preserved by contexts, we can characterize only properties which induce equivalences which are congruences. In particular we will consider some normalization properties of λ-terms, i.e. the standard properties of having a head normal form, a normal form, of being strongly normalizable (Definition 17.1.1) and that of being persistently head normalizing and persistently normalizing (Definition 17.1.2).

First we recall some basic definitions.

17.1.1. Definition (Severi [1996]). The set of *strongly normalizing terms* is the least set of terms SN closed under the following rules:

- $\dfrac{M_1{\in}\mathsf{SN},\ldots,M_n{\in}\mathsf{SN}}{xM_1\ldots M_n{\in}\mathsf{SN}}{\scriptstyle n\geq 0}$
- $\dfrac{M{\in}\mathsf{SN}}{\lambda x.M{\in}\mathsf{SN}}$
- $\dfrac{M[x:=N]M_1\ldots M_n{\in}\mathsf{SN} \qquad\qquad N{\in}\mathsf{SN}}{(\lambda xM)NM_1\ldots M_n{\in}\mathsf{SN}}{\scriptstyle n\geq 0}$

The correctness of the above definition can be easily seen using natural inductive arguments.

Persistently normalizing terms have been introduced in Böhm and Dezani-Ciancaglini [1975]. In Coppo et al. [1987] persistently normalizing normal forms have been characterized using the notion of replaceable variable (Coppo et al.

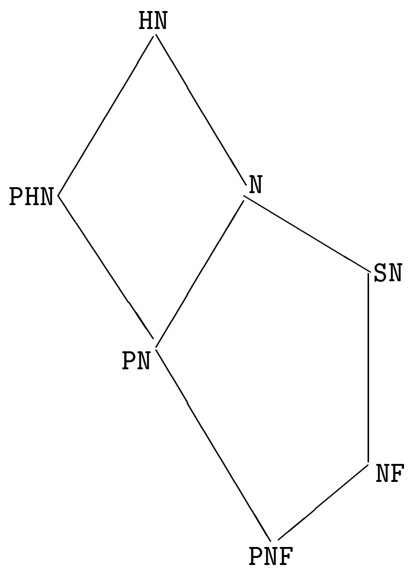

Figure 17.1: Inclusions between sets of λ-terms

[1987]). Analogously we introduce the notion of persistently head normalizing terms.

17.1.2. Definition. (i) A term M is persistently head normalizing iff $M\vec{N}$ has a head normal form for all terms $\vec{N}$.

(ii) A term M is persistently normalizing iff $M\vec{N}$ has a normal form for all normalizable terms $\vec{N}$.

The following predicates will come in handy in the sequel:

17.1.3. Definition.

(i) $M \in \mathsf{HN} \Leftrightarrow M$ has a head normal form.
(ii) $M \in \mathsf{N} \Leftrightarrow M$ has a normal form.
(iii) $M \in \mathsf{NF} \Leftrightarrow M$ is a normal form.
(iv) $M \in \mathsf{HPN} \Leftrightarrow M$ is persistently head normalizing.
(v) $M \in \mathsf{PN} \Leftrightarrow M$ is persistently normalizing.
(vi) $M \in \mathsf{PNF} \Leftrightarrow M$ is a persistently normalizing normal form.
(vii) $M \in \mathsf{SN} \Leftrightarrow M$ is strongly normalizing.

The following inclusions follow immediately by definition:
$\mathsf{PNF} \subseteq \mathsf{NF} \subseteq \mathsf{SN} \subseteq \mathsf{N}$
$\mathsf{PNF} \subseteq \mathsf{PN} \subseteq \mathsf{N} \subseteq \mathsf{HN}$
$\mathsf{PN} \subseteq \mathsf{PHN} \subseteq \mathsf{HN}$.
The above inclusions are better visualized in Figure 17.1.

We need a further class of type theories.

17.1.4. DEFINITION. A type theory $\nabla^{\triangledown}$ is an *arrow* type theory if $\Omega\in\mathbb{C}^{\triangledown}$ and for all $\psi\in\mathbb{C}^{\triangledown}$ either $\psi \sim_{\triangledown} \Omega$ or $\exists I \neq \emptyset, \{A_i, B_i\}_{i\in I}.\psi\sim_{\triangledown} \bigcap_{i\in I}(A_i\to B_i)$.

It turns out that all and only the λ-terms enjoying these properties have types of a particular shape for suitable choices of type theories, as stated in the following theorem:

17.1.5. THEOREM. *(Characterization)*

(i) *(head normalizing terms) A λ-term $M\in\mathsf{HN}$ if and only if for all non trivial type theories $\nabla^{\triangledown}$ such that $\Omega\in\mathbb{C}^{\triangledown}$, there exist $A\in\mathbb{T}^{\triangledown}$ and a $\triangledown$-basis Γ such that $\Gamma \vdash^{\triangledown}_{\Omega} M : A$ and $A\not\sim_{\triangledown}\Omega$. Moreover in the system $\lambda\cap^{BCD}_{\Omega}$ the terms satisfying the latter property are precisely the head normalizing ones.*

(ii) *(normalizing terms) A λ-term $M\in\mathsf{N}$ if and only if for all non trivial type theories $\nabla^{\triangledown}$ such that $\Omega\in\mathbb{C}^{\triangledown}$, there exist $A\in\mathbb{T}^{\triangledown}$ and a $\triangledown$-basis Γ such that $\Gamma \vdash^{\triangledown}_{\Omega} M : A$ and $\Omega \notin A, \Gamma$. Moreover in the system $\lambda\cap^{BCD}_{\Omega}$ the terms satisfying the latter property are precisely the normalizing ones. Lastly in the system $\lambda\cap^{CDZ}_{\Omega}$ the terms typable with φ in the context all whose predicates are ω are precisely the normalizing ones.*

(iii) *(strongly normalizing terms) A λ-term $M\in\mathsf{SN}$ if and only if for all type theories $\nabla^{\triangledown}$ there exist $A\in\mathbb{T}^{\triangledown}$ and a $\triangledown$-basis Γ such that $\Gamma \vdash^{\triangledown}_{\mathcal{N}} M : A$. Moreover in the system $\lambda\cap^{CDV}_{\mathcal{N}}$ the terms satisfying the latter property are precisely the strongly normalizing ones.*

(iv) *(persistently head normalizing terms) A λ-term $M\in\mathsf{PHN}$ if and only if for all type theories $\nabla^{\triangledown}$ such that $\Omega\in\mathbb{C}^{\triangledown}$ and all $A\in\mathbb{T}^{\triangledown}$ there exists a $\triangledown$-basis Γ and an integer n such that $\Gamma \vdash^{\triangledown}_{\Omega} M : \Omega^n \to A$. Moreover in the system $\lambda\cap^{Scott}_{\Omega}$ the terms typable with type ω in the context all whose predicates are ω are precisely the persistently head normalizing ones.*

(v) *(persistently normalizing terms) A λ-term $M\in\mathsf{PN}$ if and only if for all arrow-type theories $\nabla^{\triangledown}$ and all $A\in\mathbb{T}^{\triangledown}$ there exists a $\triangledown$-basis Γ such that $\Gamma \vdash^{\triangledown}_{\Omega} M : A$. Moreover in the system $\lambda\cap^{CDZ}_{\Omega}$ the terms typable with type ω in the context all whose predicates are ω are precisely the persistently normalizing ones.*

The proofs of the only if parts of the above Theorem are quite easy and follow, but for point (v), which is proved in the subsection "Polarized normal forms". The proofs of the if parts require the notion of type stable sets and are done in subsection "Stable sets".

PROOF OF THEOREM 17.1.5 ($\Leftarrow$).
(i). By Corollary 15.2.8(**??**) it suffices to consider M in hnf. Let $M \equiv \lambda\vec{y}.x\vec{M}$ where $\vec{y} \equiv y_1 \dots y_n$ and $\vec{M} \equiv M_1 \dots M_m$. We say that n is the length of $\vec{y}$ and m is the length of $\vec{M}$. We have $x{:}\Omega^m\to A \vdash^{\triangledown} x\vec{M} : A$. If x is free this implies $x{:}\Omega^m\to A \vdash^{\triangledown} M : \Omega^n\to A$. If x is bound, say $x \equiv y_i$ for some $i \leq n$, then $\vdash^{\triangledown} M : \Omega^{i-1}\to(\Omega^m\to A)\to\Omega^{n-i}\to A$.
(ii). As in the proof of (i), we can consider M in nf. The proof is by induction on M. The only interesting case is $M \equiv x\vec{M}$ where $\vec{M} \equiv M_1 \dots M_m$. By induction we have $\Gamma_j \vdash^{\triangledown} M_j : A_j$, for some Γ_j, A_j not containing Ω and for $j \leq m$.

This implies: $\uplus_{j\leq m}\Gamma_j \uplus \{x{:}A_1{\to}\ldots{\to}A_m{\to}A\} \vdash^{\triangledown} x\vec{M} : A$.
For $\lambda\cap^{\rm CDZ}_{\Omega}$ let $\Gamma = \{x{:}\omega \mid x{\in}FV(M)\}$. If $M \equiv x\vec{M}$ then by induction we have $\Gamma \vdash^{\rm CDZ}_{\Omega} M_j : \varphi$ and this implies $\Gamma \vdash^{\rm CDZ}_{\Omega} x\vec{M} : \omega$, since $\omega \sim_{\rm CDZ} \varphi \to \omega$. By rule $(\leq_{\rm CDZ})$ we conclude $\Gamma \vdash^{\rm CDZ}_{\Omega} M : \varphi$. If $M \equiv \lambda y.N$ then by induction we have $\Gamma, y : \omega \vdash^{\rm CDZ}_{\Omega} N : \varphi$ and this implies $\Gamma \vdash^{\rm CDZ}_{\Omega} M : \omega \to \varphi$. By rule $(\leq_{\rm CDZ})$ we conclude $\Gamma \vdash^{\rm CDZ}_{\Omega} M : \varphi$.
(iii). By induction on the structure of strongly normalizing terms (see Definition 17.1.1). The only case not considered in the proof of (ii) is $M \equiv (\lambda x.R)N\vec{M}$ where $\vec{M} \equiv M_1 \ldots M_m$ and both $R[x := N]\vec{M}$ and N are strongly normalizing. By induction hypothesis there are Γ, A, Γ', B such that $\Gamma \vdash^{\triangledown} R[x := N]\vec{M} : A$ and $\Gamma' \vdash^{\triangledown} N : B$. We get $\Gamma \uplus \Gamma' \vdash^{\triangledown} R[x := N]\vec{M} : A$ and $\Gamma \uplus \Gamma' \vdash^{\triangledown} N : B$, so if $m = 0$ we are done by Theorem 15.2.5(**??**). If $m > 0$ by iterated applications of Lemma 15.1.2(**??**) to $\Gamma \vdash^{\triangledown} R[x := N]\vec{M} : A$ we have
$\Gamma \vdash^{\triangledown} R[x := N] : B^{(i)}_1{\to}\ldots{\to}B^{(i)}_m{\to}B^{(i)}$, $\Gamma \vdash^{\triangledown} M_j : B^{(i)}_j$ $(j \leq m)$ and $\bigcap_{i\in I} B^{(i)} \leq_{\triangledown} A$ for some $I, B^{(i)}_j (j \leq m), B^{(i)}{\in}\mathbb{T}^{\triangledown}$.
As in case $m = 0$ we obtain $\Gamma \uplus \Gamma' \vdash^{\triangledown} (\lambda x.R)N : B^{(i)}_1{\to}\ldots{\to}B^{(i)}_m{\to}B^{(i)}$. So we can conclude $\Gamma \uplus \Gamma' \vdash^{\triangledown} (\lambda x.R)N\vec{M} : A$.
(iv). If M is persistently head normalizing then the head variable of the head normal form of M must be free. In fact if $M = \lambda\vec{x}y\vec{z}.y\vec{N}$ we get $M\vec{X}(\Delta\Delta) \to \lambda\vec{z}.\Delta\Delta\vec{N'}$ (where $\vec{X}$ has the same lenght of $\vec{x}$ and $\vec{N'} = \vec{N}[\vec{x} := \vec{X}, y := \Delta\Delta]$) which does not have a head normal form. We can type a term of the shape $\lambda\vec{x}.y\vec{N}$ where $y \notin \vec{x}$ as follows $y : \Omega^m \to A \vdash^{\triangledown} \lambda\vec{x}.y\vec{N} : \Omega^n \to A$, where m is the length of $\vec{N}$ and n is the length of $\vec{x}$. ∎

17.1.6. Remark. There are many proofs of Point (iii) in the literature Pottinger [1981], Leivant [1986], van Bakel [1992], Krivine [1990], Ghilezan [1996], Amadio and Curien [1998]. As observed in Venneri [1996] all but Amadio and Curien [1998] contain some bugs, which in the case of Krivine [1990] can be easily remedied with a suitable non-standard notion of length of reduction path.

Polarized normal forms

We need to introduce the notion of polarity for term variables.

17.1.7. Definition (Polarized normal forms). Assume that the variables of λ-calculus are partitioned into two infinite sets of positive and negative variables, i.e. x^+ or x^-. Let $\Lambda^{\pm}$ be the resulting language of polarized λ-terms, and define the set of *polarized normal forms*, $\mathsf{NF}^{\mathsf{i,j}}$ as follows

$$\frac{\vec{M}{\in}\mathsf{NF}^{+,-}}{x^+\vec{M}{\in}\mathsf{NF}^{+,+} \cap \mathsf{NF}^{-,+}}\,(+\,app) \qquad \frac{\vec{M}{\in}\mathsf{NF}^{-,+} \cup \mathsf{NF}^{-,-}}{x^-\vec{M}{\in}\mathsf{NF}^{+,-} \cap \mathsf{NF}^{-,-}}\,(-\,app)$$

$$\frac{M{\in}\mathsf{NF}^{+,\mathsf{j}}}{\lambda x^+.M{\in}\mathsf{NF}^{+,\mathsf{j}}}\,(+\,abs) \qquad \frac{M{\in}\mathsf{NF}^{-,\mathsf{j}}}{\lambda x^-.M{\in}\mathsf{NF}^{-,\mathsf{j}}}\,(-\,abs)\ ∎$$

Rule $(+\ app)$ says that we can apply a positive variable only to normal forms whose initial bound variables are positive and whose head variable is negative. The so obtained normal form belongs both to $\mathsf{NF}^{+,+}$ and $\mathsf{NF}^{-,+}$: in fact it is λ-free and his head variable is positive. Rule $(-\ app)$ allows to apply a negative variable to all normal forms whose initial bound variables are negative independently from the polarity of the head variables. The rules for abstractions force all consecutive abstractions to have the same sign: they preserve the sign of the head variable.

In other words, $M{\in}\mathsf{NF}^{\mathsf{i},\mathsf{j}}$ means that the variables bound in the initial abstractions of M have polarity i, the head variable of M has polarity j and the components of M belong to $\mathsf{NF}^{\mathsf{j},+}$ or to $\mathsf{NF}^{\mathsf{j},-}$ if $\mathsf{i}=-$, and to $\mathsf{NF}^{\mathsf{j},-}$ if $\mathsf{i}=+$. A λ-free normal form can belong to both $\mathsf{NF}^{+,\mathsf{j}}$ and $\mathsf{NF}^{-,\mathsf{j}}$, since we do not know the polarity of missing bound variables.

17.1.8. DEFINITION (Positive normal forms). A term M is a positive normal form ($M{\in}\mathsf{NF}^{+}$) if and only if there exists $N{\in}\mathsf{NF}^{+,-}$ such that M is obtained form N by erasing all polarities. ∎

The notion of positive normal form can be easily understood by considering the tree structure of a normal form. A normal form is positive if and only if we can decorate its variables in such a way that:
i) the free variables are negative
ii) the variables bound in the root are positive
iii) the head variable is free (negative)
iv) each variable which is bound in a node different from the root has the same polarity of the head variable at the parent node
v) if the head variables of a node is positive then the head variables of all sons of this node must be negative.

We can prove that polarized normal forms can be typed starting from arbitrary types for positive variables, when we restrict to arrow type theories. Moreover under the same hypothesis the polarized normal forms corresponding to positive normal forms can be typed with an arbitrary type.

17.1.9. THEOREM. *Let $\nabla^{\triangledown}$ be an arrow type theory. Let $M{\in}\mathsf{NF}^{\mathsf{i},\mathsf{j}}$ and let $\vec{x}^{+}$ and $\vec{y}^{-}$ be the positive and negative variables in M:*

(i) *if $\mathsf{i}=+$ and $\mathsf{j}=-$ then for all types $\vec{A}$ assigned to positive variables $\vec{x}^{+}$ and for all types A there exist types $\vec{B}$ such that $\vec{x^{+}:A},\vec{y^{-}:B}\vdash^{\triangledown} M:A$.*

(ii) *otherwise for all types $\vec{A}$ assigned to positive variables $\vec{x}^{+}$ there exist types $\vec{B}$ and a type A such that $\vec{x^{+}:A},\vec{y^{-}:B}\vdash^{\triangledown} M:A$.*

PROOF. We prove (i) and (ii) simultaneously by induction on the structure of polarized normal forms.

(i) If $M{\in}\mathsf{NF}^{+,-}$ then M is of the shape $\lambda\vec{z}^{+}.y^{-}\vec{N}$ where $\vec{N}{\in}\mathsf{NF}^{-,+}\cup\mathsf{NF}^{-,-}$. Since $\nabla^{\triangledown}$ is an arrow type theory, then each type is equivalent either to Ω or to an intersection of arrow types. So it sufficies to prove that M has all types of the shape $\vec{C}\to D$, where $\vec{C}$ has the length of $\vec{z}$. By the induction hypothesis (ii) there are types $\vec{B}$ and $\vec{E}$ such that for all types $\vec{A}$ and $\vec{C}$ we have: $\vec{x^{+}:A},\vec{z^{+}:C},\vec{y^{-}:B}\vdash^{\triangledown}\vec{N}:\vec{E}$. Now let Γ be the basis obtained by adding the premise $y^- : \vec{E} \to D$ to $x^+ \vec{:} A, z^+ \vec{:} C, y^- \vec{:} B$. We get $\Gamma \vdash^\nabla y^-\vec{N} : D$ and we can conclude using rule $(\to \mathrm{I})$.

(ii) If $M \in \mathsf{NF}^{-,-}$ then M is of the shape $\lambda \vec{z^-}.t^-\vec{N}$ where $\vec{N} \in \mathsf{NF}^{-,+} \cup \mathsf{NF}^{-,-}$ and $t^- \in \vec{y^-} \cup \vec{z^-}$. By the induction hypothesis (ii) there are types $\vec{B}, \vec{C}$ and $\vec{E}$ such that for all types $\vec{A}$: $x^+ \vec{:} A, z^- \vec{:} C, y^- \vec{:} B \vdash^\nabla \vec{N} : \vec{E}$. Now let Γ be the basis obtained by adding the premise $t^- : \vec{E} \to D$, where D is arbitrary, to $x^+ \vec{:} A, z^- \vec{:} C, y^- \vec{:} B$. We get $\Gamma \vdash^\nabla t^-\vec{N} : D$ and we can conclude using rule $(\to \mathrm{I})$.
If $M \in \mathsf{NF}^{-,+}$ then M is of the shape $\lambda \vec{z^-}.x^+\vec{N}$ where $\vec{N} \in \mathsf{NF}^{+,-}$. Let A' be the type assigned to the variable x^+. Since ∇^∇ is an arrow-type theory, we can find arrow types bigger than or equal to an arbitrary type. So let $A' \leq_{\nabla} {}_{\nabla}\vec{E} \to D$, where $\vec{E}$ has the length of $\vec{N}$. By the induction hypothesis (i) there are types $\vec{B}$ and $\vec{C}$ such that for all types $\vec{A}$ and $\vec{E}$: $x^+ \vec{:} A, z^- \vec{:} C, y^- \vec{:} B \vdash^\nabla \vec{N} : \vec{E}$. We get $x^+ \vec{:} A, z^- \vec{:} C, y^- \vec{:} B \vdash^\nabla x^+\vec{N} : D$ and we can conclude using rule $(\to \mathrm{I})$.
If $M \in \mathsf{NF}^{+,+}$ then M is of the shape $\lambda \vec{z^+}.t^+\vec{N}$ where $\vec{N} \in \mathsf{NF}^{+,-}$ and $t^+ \in \vec{x^+} \cup \vec{z^+}$. If $t^+ \in \vec{x^+}$ the proof goes as in previous case. Otherwise the proof is similar, since we can assume $t^+ : \vec{E} \to D$, where $\vec{E}$ has the length of $\vec{N}$, for arbitrary types $\vec{E}, D$ and conclude as in previous case. ■

We give now some definitions concerning only terms in normal form: we assume that all free and bound variables have different names. Moreover we do forbid α-conversion. This allow us to consider meaningfull also the names of bound variables.

We consider here again positive and negative variables, but we extend the set of polarized normal forms by modifying rule $(+\ app)$ as follows:

$$\frac{\vec{M} \in \mathsf{NF}^{+,+} \cup \mathsf{NF}^{+,-}}{x^+\vec{M} \in \mathsf{NF}^{+,+} \cap \mathsf{NF}^{-,+}}\ (+\ app')$$

It is easy to check that each normal form correspond to a polarized one in this extended set. So we can assign polarities to variables in an arbitrary normal form, starting from the conventions that free variables are negative and variables bound in the initial abstractions are positive. We recall first some standard definitions.

17.1.10. Definition. (i) In a normal form of the shape $x\vec{M}(\lambda \vec{z}.y\vec{N})$ we say that the showed occurrences of x and y are *adjacent*. Two variables are *adjacent* in a term M iff they have adjacent occurrences in some subterm of M.

(ii) If $M \equiv xN_1 \ldots N_i \ldots N_m$ we say that the subterm N_i is the i-th argument of x in M.

(iii) If $M \equiv \lambda y_1 \ldots y_j \ldots y_n.N$ we say that the variable y_j is the j-th abstraction of M. ■

We need to introduce the replacement path of a negative variable in a normal form. Intuitively this says us if the variable is free or bound, and in the last case where it is bound. This is useful in order to replace that variable using substitutions of free variables.

The replacement path of a free variable is the variable itself. The replacement path of a negative bound variable is the name of a free variable followed by a sequence of integer pairs. If the replacement path of the variable y in the normal form M is $x\langle i_1, j_1\rangle \dots \langle i_n, j_n\rangle$, then there is an occurrence of x in M such that if z_1 is the j_1-abstraction of the i_1 argument of x, then there is an occurrence of z_1 in M such that if z_2 is the j_2-abstraction of the i_2 argument of z_1, then ... there is an occurrence of z_{n-1} in M such that y is the j_n-abstraction of the i_n argument of z_{n-1}. A constructive definition is as follows.

17.1.11. DEFINITION. The *replacement path* $\pi(x, M)$ of a variable x in a normal form M is inductively defined by:

$$\pi(x, M) = x \text{ if } x \text{ is free in } M$$

$$\pi(x, y\vec{N}M\vec{N}') = \pi(x, M) \text{ if } \pi(x, M) \text{ is defined}$$

$$\frac{\pi(x, M) = y\alpha \text{ and } z \neq y}{\pi(x, \lambda z.M) = \pi(x, M)}$$

$$\frac{\pi(x, M) = y_j\alpha}{\pi(x, zN_1 \dots N_{i-1}(\lambda y_1 \dots y_j \dots y_n.M)N_{i+1} \dots N_m) = z\langle i, j\rangle\alpha} \blacksquare$$

For example if $M \equiv \lambda v.x(\lambda t.a(t(\lambda uz.b(xv(\lambda y.d(yz))))))$ then $\pi(x, M) = x$, $\pi(y, M) = x\langle 2, 1\rangle$, $\pi(z, M) = x\langle 1, 1\rangle\langle 1, 2\rangle$, while $\pi(v, M)$ is undefined.

It is easy to verify that all and only the negative variables have defined replacement paths.

Notice that the replacement path of a variable can be undefined in a subterm and defined in the whole term. This comes from the last clause of Definition 17.1.11. In the above example $\pi(y, \lambda y.d(yz))$ is undefined, while $\pi(y, M) = x\langle 2, 1\rangle$.

17.1.12. LEMMA. *If z, t are adjacent in a normal form M with replacements paths $x\alpha$, $y\beta$, then there are normal forms X, Y such that $M[x := X, y := Y]$ does not have a normal form (possibly $x \equiv y$ and, in this case, $X \equiv Y$).*

PROOF. Actually ve prove a stronger statement, i.e. we require X and Y to be λ-terms in which all abstracted variables occur at least once as arguments of a free variable. This makes sure that all subterms of all terms obtained out of $M[x := X, y := Y]$ by reduction will never be erased. So we need only build a reduct of $M[x := X, y := Y]$ containing an unsolvable subterm.
The proof is by induction on the sum of the lengths of α and β. We convene that (the considered occurrence of) z is on the left of (the considered occurrence of) t in M.

Basic step. In this case α and β are empty, and therefore $x \equiv z$ and $y \equiv t$. By definition M has a subterm of the shape
$xN_1 \dots N_{i-1}(\lambda u_1 \dots u_n.yN'_1 \dots N'_m)$.

If $x \not\equiv y$ a possible choice is

$X \equiv \lambda v_1 \ldots v_i.av_1 \ldots v_i(v_iu_1 \ldots u_n\Delta)$ and $Y \equiv \lambda w_1 \ldots w_mw.bw_1 \ldots w_mw(ww)$,

where $\Delta \equiv \lambda w.ww$. We leave to the reader to check that $M[x := X, y := Y]$ will never reduces to a normal form.

If $x \equiv y$ we can choose

$$X \equiv \lambda v_1 \ldots v_k.av_1 \ldots v_k(v_{m+1}v_{m+1})(v_iu_1 \ldots u_n(v_iu_1 \ldots u_n)),$$

where k is the maximum between i and $m+1$. It is easy to verify that $xN_1 \ldots N_{i-1}(\lambda u_1 \ldots u_n.xN'_1 \ldots N'm)[x := X]$ does not have a normal form.

Induction step: we need to distinguish four possible cases. In all cases we exhibit a normal form X' such that z, t remain adjacent in the normal form M' of $M[x := X']$ but the sum of the lengths of their replacement paths is decreased. This allows us to apply the induction.

Case $x \not\equiv y$. Assume that $\alpha = \langle i, j\rangle\alpha'$. Then M has a subterm of the shape $xN_1 \ldots N_{i-1}(\lambda u_1 \ldots u_j.N')$ and $\pi(z, N') = u_j\alpha'$. Notice that $z \equiv u_j$ if α' is empty.

Let $X' \equiv \lambda v_1 \ldots v_i.xv_1 \ldots v_i(v_iu_1 \ldots u_j)$ and M' be the normal form of $M[x := X']$ (the existence of M' comes from the fact that the substitution of X' to x only applies subterms of M to free variables). On M' we can observe that:

- the variables which are adjacent in M remain adjacent in M' with the same names, and the variables whose replacement path in M was $x\langle i, l\rangle$ with $1 \leq l \leq j$ occur also with name u_l in M';
- the variables whose replacement path in M was $x\langle i, l\rangle\gamma$ with $1 \leq l \leq j$ occur also with replacement path $u_l\gamma$ in M';
- the variables whose replacement path in M was $x\langle i, l\rangle\gamma$ with $l > j$ occur also with replacement path $x\langle i, l-j\rangle\gamma$ in M';
- all other variables will have the same replacement paths in M and M'.

By the above observations z, t are adjacent in M' and $\pi(z, M') = u_j\alpha'$, $\pi(t, M') = y\beta$. Then induction hypothesis applies and we can find normal forms U_j, Y such that $M'[u_j := U_j, y := Y]$ does not reduce to a normal form. Therefore we can choose X equal to $X'[u_j := U_j]$.

Case $x \equiv y$ and α, β both not empty. Assume that $\alpha = \langle i, j\rangle\alpha'$ and $\beta = \langle n, m\rangle\beta'$. Then M has a subterm of the shape $xM_1 \ldots M_{i-1}(\lambda u_1 \ldots u_j.M^*)$ with $\pi(z, M^*) = u_j\alpha'$ and a subterm of the shape $xN_1 \ldots N_{n-1}(\lambda v_1 \ldots v_m.N^*)$ with $\pi(t, N^*) = v_m\beta'$ (these subterms can coincide). Observe that the adjacent occurences of z, t must be subterms of both M^* and N^*, and this implies that the showed occurrences of x are either nested or they coincide and $i = n$. Notice that $z \equiv u_j$ if α' is empty and $t \equiv v_m$ if β' is empty.

We first consider the subcase $i \neq n$. This implies $u_j \not\equiv v_m$. Let $X' \equiv \lambda w_1 \ldots w_k.xw_1 \ldots w_k(w_iu_1 \ldots u_j)(w_nv_1 \ldots v_m)$, where k is the maximum between i and n, and let M' be the normal form of $M[x := X']$. On M' we can

observe that:
- the variables which are adjacent in M remain adjacent in M' with the same names, and the variables whose replacement path in M was $x\langle i,l\rangle$ with $1 \leq l \leq j$ occur also with name u_l in M', and the variables whose replacement path in M was $x\langle n,h\rangle$ with $1 \leq h \leq m$ occur also with name v_h in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $1 \leq l \leq j$ occur also with replacement path $u_l\gamma$ in M';
- the variables whose replacement path in M was $x\langle n,h\rangle\gamma$ with $1 \leq h \leq m$ occur also with replacement path $v_h\gamma$ in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $l > j$ occur also with replacement path $x\langle i,l-j\rangle\gamma$ in M';
- the variables whose replacement path in M was $x\langle n,h\rangle\gamma$ with $h > m$ occur also with replacement path $x\langle n,h-m\rangle\gamma$ in M';
- all other variables will have the same replacement paths in M and M'.

By the above observations z,t are adjacent in M' and $\pi(z,M') = u_j\alpha'$, $\pi(t,M') = v_m\beta'$. Then induction hypothesis applies and we can find normal forms U_j, V_m such that $M'[u_j := U_j, v_m := V_m]$ does not reduce to a normal form.
Then we can choose X equal to $X'[u_j := U_j, v_m := V_m]$.

If $i = n$ let $X' \equiv \lambda w_1 \ldots w_n.xw_1 \ldots w_n(w_iu_1 \ldots u_k)$, where k is the maximun between j and m. On the normal form M' of $M[x := X']$ we can observe that:
- the variables which are adjacent in M remain adjacent in M' with the same names, and the variables whose replacement path in M was $x\langle i,l\rangle$ with $1 \leq l \leq k$ occur also with name u_l;
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $1 \leq l \leq k$ occur also with replacement path $u_l\gamma$ in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $l > k$ occur also with replacement path $x\langle i,l-k\rangle\gamma$ in M';
- all other variables will have the same replacement paths in M and M'.

By the above observations z,t are adjacent in M' and $\pi(z,M') = u_j\alpha'$, $\pi(t,M') = u_m\beta'$. Then induction hypothesis applies and we can find normal forms U_j, U_m such that $M'[u_j := U_j, u_m := U_m]$ (or $M'[u_j := U_j]$ if $j = m$) does not reduce to a normal form.
Then we can choose X equal to $X'[u_j := U_j, u_m := U_m]$ if $j \neq m$ and equal to $X'[u_j := U_j]$ if $j = m$.

Case $x \equiv y$ and α empty, while β not empty. Then $z \equiv x$ and $t \not\equiv x$. Assume that $\beta = \langle i,j\rangle\beta'$. Then M has a subterm of the shape $xM_1 \ldots M_{i-1}(\lambda u_1 \ldots u_j.M^*)$ with $\pi(t,M^*) = u_j\beta'$ and a subterm of the shape $xN_1 \ldots N_{k-1}(\lambda v_1 \ldots v_n.tN'_1 \ldots N'_m)$ (these subterms can coincide). Notice that $t \equiv u_j$ if β' is empty.
Let $X' \equiv \lambda w_1 \ldots w_i.xw_1 \ldots w_i(w_iu_1 \ldots u_j)$ and let M' be the normal form of $M[x := X']$. On M' we can observe that:
- the variables which are adjacent in M remain adjacent in M' with the same names, and the variables whose replacement path in M was $x\langle i,l\rangle$ with $1 \leq l \leq j$ occur also with name u_l in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $1 \leq l \leq j$ occur

also with replacement path $u_l\gamma$ in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $l > j$ occur also with replacement path $x\langle i,l-j\rangle\gamma$ in M';
- all other variables will have the same replacement paths in M and M'.

By the above observations x, t are adjacent in M' and $\pi(t, M') = u_j\beta'$. Then induction hypothesis applies and we can find normal forms X'', U_j such that $M'[x := X'', u_j := U_j]$ does not reduce to a normal form.
Notice that X'' will be build in the basic step, so X'' will be equal to $\lambda w_1 \dots w_k.aw_1 \dots w_k(w_k v_1 \dots v_n \Delta)$.
If the subterms $xM_1 \dots M_{i-1}(\lambda u_1 \dots u_j.M^*)$ and $xN_1 \dots N_{n-1}(\lambda v_1 \dots v_m.N^*)$ do not coincide then the second one must be a subterm of M^*. We can choose X equal to
$\lambda w_1 \dots w_h.aw_1 \dots w_h(w_i u_1 \dots u_{j-1}U_j)(w_k v_1 \dots v_n \Delta)$ where h is the maximum between i and k.
To see why this works, observe that a subterm of (a reduct of) $M^*[x := X]$ will be $xN_1 \dots N_{k-1}(\lambda v_1 \dots v_n.tN'_1 \dots N'_m)[x := X, t := T]$, where T is a subterm of U_j. Notice that T is build in the basic step, and therefore $T \equiv \lambda r_1 \dots r_m r.br_1 \dots r_m r(rr)$. Now if $R \equiv \lambda v_1 \dots v_n.T\hat{N}'_1 \dots \hat{N}'_m$ we get
$X\hat{N}_1 \dots \hat{N}_{k-1}R \to \lambda w_{k+1} \dots w_h.a\hat{N}_1 \dots \hat{N}_{k-1}Rw_{k+1} \dots w_h S(Rv_1 \dots v_n \Delta)$
and $Rv_1 \dots v_n\Delta \to b\hat{N}'_1 \dots \hat{N}'_m\Delta(\Delta\Delta)$, where $\hat{N} = N[x := X, t := T]$ (possibly with indeces and $'$) and S is a suitable term.
If the subterms $xM_1 \dots M_{i-1}(\lambda u_1 \dots u_j.M^*)$ and $xN_1 \dots N_{n-1}(\lambda v_1 \dots v_m.N^*)$ coincide we get $i = k$, $t \equiv u_j$ and $j \geq n$. In this case we can choose X equal to $\lambda w_1 \dots w_i.aw_1 \dots w_i(w_i u_1 \dots u_{j-1}U_j u_{j+1} \dots u_n \Delta)$.
Notice that U_j is build in the basic step, and therefore $U_j \equiv \lambda r_1 \dots r_m r.br_1 \dots r_m r(rr)$.
To see why this works, observe that a subterm of (a reduct of) $M[x := X]$ will be $xN_1 \dots N_{i-1}(\lambda u_1 \dots u_n.u_j N'_1 \dots N'_m)[x := X]$.
Now we get
$X\hat{N}_1 \dots \hat{N}_{i-1}(\lambda u_1 \dots u_n.u_j\hat{N}'_1 \dots \hat{N}'_m) \to$
$a\hat{N}_1 \dots \hat{N}_{i-1}(\lambda u_1 \dots u_n.u_j\hat{N}'_1 \dots \hat{N}'_m)(U_j\hat{\hat{N}}'_1 \dots \hat{\hat{N}}'_m\Delta)$
and $U_j\hat{\hat{N}}'_1 \dots \hat{\hat{N}}'_m\Delta \to b\hat{\hat{N}}'_1 \dots \hat{\hat{N}}'_m\Delta(\Delta\Delta)$, where $\hat{N} = N[x := X]$ and $\hat{\hat{N}} = \hat{N}[u_j := U_j]$ (possibly with indeces and $'$).

Case $x \equiv y$ and α not empty, while β empty. Then $z \not\equiv x$ and $t \equiv x$. Assume that $\alpha = \langle i,j\rangle\alpha'$. Then M has a subterm of the shape $xM_1 \dots M_{i-1}(\lambda u_1 \dots u_j.M^*)$ with $\pi(z, M^*) = u_j\alpha'$ and M^* has a subterm of the shape
$zN_1 \dots N_{k-1}(\lambda v_1 \dots v_n.xN'_1 \dots N'_m)$. Notice that $z \equiv u_j$ if α' is empty.
Let h be the maximum between i and $m+1$ (so $h \geq 1$) and
$X' \equiv \lambda w_1 \dots w_h.xw_1 \dots w_{h-1}(xw_h(w_i u_1 \dots u_j))$. On the normal form M' of $M[x := X']$ we can observe that:
- the variables which are adjacent in M remain adjacent in M' with the same names, and the variables whose replacement path in M was $x\langle i,l\rangle$ with $1 \leq l \leq j$ occur also with name u_l in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $1 \leq l \leq j$ occur also with replacement path $u_l\gamma$ in M';
- the variables whose replacement path in M was $x\langle i,l\rangle\gamma$ with $l > j$ occur also

with replacement path $x\langle i, l-j\rangle\gamma$ in M';
- all other variables will have the same replacement paths in M and M'.

By the above observations z, x are adjacent in M' and $\pi(z, M') = u_j\alpha'$. Then induction hypothesis applies and we can find normal forms X'', U_j such that $M'[x := X'', u_j := U_j]$ does not reduce to a normal form.
Notice that $\lambda v_1 \ldots v_n.xN'_1 \ldots N'_m[x := X']$ reduces to a normal form of the shape
$\lambda v_1 \ldots v_n w_{m+1} \ldots w_h.xR_1 \ldots R_h$ for suitable normal forms $R_1, \ldots, R_h$. Since X'' will be build in the basic step, it will be equal to
$\lambda r_1 \ldots r_{h+1}.br_1 \ldots r_{h+1}(r_{h+1}r_{h+1})$. This suggests us to choose X equal to
$\lambda s_1 \ldots s_{h+1}.xs_1 \ldots s_{h-1}(xs_h s_{h+1}(s_i u_1 \ldots U_j)(s_{h+1}s_{h+1}))$.
To see why this works, observe that a subterm of (a reduct of) $M[x := X]$ will be $zN_1 \ldots N_{k-1}(\lambda v_1 \ldots v_n.xN'_1 \ldots N'_m)[x := X, z := Z]$ where Z is a subterm of U_j. Notice that Z is build in the basic step, and therefore $Z \equiv \lambda q_1 \ldots q_k.aq_1 \ldots q_k(q_k v_1 \ldots v_n w_{m+1} \ldots w_h\Delta)$. Now if $R \equiv (\lambda v_1 \ldots v_n.X\hat{N}'_1 \ldots \hat{N}'_m)$
$Z\hat{N}_1 \ldots \hat{N}_{k-1}R \to a\hat{N}_1 \ldots \hat{N}_{k-1}R(Rv_1 \ldots v_n w_{m+1} \ldots w_h\Delta)$
and $Rv_1 \ldots v_n w_{m+1} \ldots w_h\Delta \to x\hat{N}'_1 \ldots \hat{N}'_m w_{m+1} \ldots w_{h-1}(xw_h\Delta S(\Delta\Delta))$, where $\hat{N} = N[x := X, t := Z]$ (possibly with indeces and $'$) and S is a suitable term.

Notice that in all cases X is a normal form. ■

17.1.13. THEOREM. *All persistently normalizing normal forms are positive normal forms, i.e.* $\mathsf{PNF} \subseteq \mathsf{NF}^+$.

PROOF. We show that if M does not belong to NF^+ then M does not belong to PNF. The easier case is when M does not belong to NF^+ since the head variable of M is bound. Let $M \equiv \lambda\vec{x}y\vec{z}.y\vec{N}$. Then clearly $M\vec{x}(\lambda\vec{t}u.u\Delta\vec{t})\vec{z}\Delta \to \Delta\Delta\vec{N}'$ where $\vec{t}$ has the same length as $\vec{N}$ and $N' \equiv N[y := \lambda\vec{t}u.u\Delta\vec{t}]$.

Otherwise there must be two positive variables in M which are adjacent. Let $M \equiv \lambda\vec{x}.a\vec{N}$. All positive variables of M are negative variables of $a\vec{N}$ and their replacement paths in $a\vec{N}$ start with one variable belonging to $\vec{x}$. Therefore by Lemma 17.1.12 there are normal forms $\vec{X}$ such that $a\vec{N}[\vec{x} := \vec{X}]$ does not reduce to normal form. Therefore also $M\vec{X}$ does not reduce to normal form and this implies $M \notin \mathsf{PNF}$. ■

17.1.14. EXAMPLE. If M is $x(\lambda y.a(y(\lambda z.b(x(\lambda t.c(zt))))))$, then the variables z, t are adjacent in $\lambda x.M$ and their replacement paths in M are respectively $x\langle 1,1\rangle\langle 1,1\rangle$ and $x\langle 1,1\rangle$. Following the proof of Lemma 17.1.12 we can consider the normal form M' of $M[x := X']$ where $X' \equiv \lambda u.xu(uy)$. We get $M' \equiv x(\lambda y.a(y(\lambda z.b(x(\lambda t.c(zt))(c(zy))))))(a(y(\lambda z.b(x(\lambda t.c(zt))(c(zy))))))$. The variables z, y are adjacent in M' with replacement paths respectively $y\langle 1,1\rangle$ and y. Now the normal form M'' of $M'[y := Y']$, where $Y' \equiv \lambda v.y(yv(vz))$, is
$x(\lambda y.a(y(\lambda z.b(x(\lambda t.c(zt))(c(zy))))))(a(y(y(\lambda z.b(x(\lambda t.c(zt))(c(zy))))$
$(b(y(\lambda t.c(zt))(c(zY'))))))$. In M'' the variable z, y are adjacent with replacement paths z, y. So if we choose $Z \equiv \lambda w.a(wv\Delta)$, $Y'' \equiv \lambda r_1 r_2.br_1(r_2 r_2)$ we get $M''[z := Z, y := Y'']$ which does not have a normal form. Now following the proof we build $Y \equiv \lambda s_1 s_2.y(ys_1 s_2(s_1 Z)(s_2 s_2))$ and also $M''[y := Y]$ does

not have a normal form. Lastly replacing Y to y in X' we obtain the term $X \equiv \lambda u.xu(uY)$ and one can check that the application of M to X does not have a normal form. The crucial steps are:
$MX \to XN_1 \to xN_1(N_1Y)$ where $N_1 \equiv \lambda y.a(y(\lambda z.b(X(\lambda t.c(zt)))))$
$N_1Y \to a(YN_2)$ where $N_2 \equiv \lambda z.b(X(\lambda t.c(zt)))$
$YN_2 \to \lambda s_2.y(yN_2s_2(N_2Z)(s_2s_2))$
$N_2Z \to b(X(\lambda t.c(Zt)))$
$X(\lambda t.c(Zt)) \to xN_4(N_4Y)$ where $N_4 \equiv \lambda t.c(Zt)$
$N_4Y \to c(ZY) \to c(a(Yv\Delta))$
$Yv\Delta \to y(yv\Delta(vZ)(\Delta\Delta))$.

Proof of 17.1.5(v) **($\Leftarrow$)**. If $M \in \mathsf{PN}$ then by definition its normal form $M' \in \mathsf{PNF}$. By Theorem 17.1.13 $M' \in \mathsf{NF}^+$, so M' has all types in an arbitrary arrow type theory by Theorem 17.1.9. We can conclude that also M has all types in an arbitrary arrow type theory by Theorem 15.2.5(**??**). ■

Stable sets

We will use the standard proof technique of type stable sets (Krivine [1990]).

17.1.15. Definition. (i) A set $X \subseteq \mathsf{HN}$ is HN-*type-stable* if it contains $x\vec{M}$ for all $\vec{M} \in \Lambda$ and it is closed under head expansion of redexes;
(ii) A set $X \subseteq \mathsf{N}$ is N-*type-stable* if it contains $x\vec{M}$ for all $\vec{M} \in \mathsf{N}$ and it is closed under head expansion of redexes;
(iii) A set $X \subseteq \mathsf{SN}$ is SN-*type-stable* if it contains $x\vec{M}$ for all $\vec{M} \in \mathsf{SN}$ and it is closed under head expansion of λ-**I**-redexes or of λ-**K**-redexes whose argument is in SN. ■

From the above definition we easily get:

17.1.16. Proposition. *Let* $\mathsf{S} \in \{\mathsf{HN}, \mathsf{N}, \mathsf{SN}\}$ *and* $X \Rightarrow Y = \{M \in \Lambda \mid \forall N \in X\ MN \in Y\}$.

(i) *Each* S *is* S-*type-stable.*
(ii) *If* X, Y *are* S-*type-stable then* $X \Rightarrow Y$ *is* S-*type-stable.*
(iii) *If* Y *is* HN-*type-stable then* $\Lambda \Rightarrow Y$ *is* HN-*type-stable.*
(iv) *If* X, Y *are* S-*type-stable then* $X \cap Y$ *is* S-*type-stable.* ■

We need to adapt the definitions of type interpretation and semantic satisfability from Section 17.2 to the present context.

17.1.17. Definition. (Type Interpretation) The *interpretation* of types in $\mathbb{T}^{\triangledown}$ induced by the type environment $\mathcal{V} : \mathbb{C}^{\triangledown} \to \mathcal{P}(\Lambda)$ is defined by:
(i) $[\![A]\!]_{\mathcal{V}} = \mathcal{V}(A)$ if $A \in \mathbb{C}^{\triangledown}$;
(ii) $[\![A \to B]\!]_{\mathcal{V}} = [\![A]\!]_{\mathcal{V}} \Rightarrow [\![B]\!]_{\mathcal{V}}$;
(iii) $[\![A \cap B]\!]_{\mathcal{V}} = [\![A]\!]_{\mathcal{V}} \cap [\![B]\!]_{\mathcal{V}}$. ■

For the reader's sake, we recall from Section 17.2 that a type environment $\mathcal{V}$ *agrees* with a type theory $\nabla^\triangledown$ iff

$$A \leq_\triangledown B \;\Rightarrow\; [\![A]\!]_\mathcal{V} \subseteq [\![B]\!]_\mathcal{V}.$$

We define some type environments which agree with the different type theories considered in Theorem 17.1.5.

17.1.18. Definition. (Type Interpretation)
(i) The *type environment* $\mathcal{V}^1_{\mathrm{BCD}}$ is defined by:
$[\![A]\!]_\mathcal{V} = \mathsf{HN}$ if $A \in \mathbb{C}_\infty$;
$[\![\Omega]\!]_\mathcal{V} = \Lambda$.
(ii) The *type environment* $\mathcal{V}^2_{\mathrm{BCD}}$ is defined by:
$[\![A]\!]_\mathcal{V} = \mathsf{N}$ if $A \in \mathbb{C}_\infty$;
$[\![\Omega]\!]_\mathcal{V} = \Lambda$.
(iii) The *type environment* $\mathcal{V}_{\mathrm{CDZ}}$ is defined by:
$[\![\omega]\!]_\mathcal{V} = \mathsf{PN}$;
$[\![\varphi]\!]_\mathcal{V} = \mathsf{N}$;
$[\![\Omega]\!]_\mathcal{V} = \Lambda$.
(iv) The *type environment* $\mathcal{V}_{\mathrm{CDV}}$ is defined by:
$[\![A]\!]_\mathcal{V} = \mathsf{SN}$ if $A \in \mathbb{C}_\infty$.
(v) The *type environment* $\mathcal{V}_{\mathrm{Scott}}$ is defined by:
$[\![\omega]\!]_\mathcal{V} = \mathsf{PHN}$;
$[\![\Omega]\!]_\mathcal{V} = \Lambda$. ■

It is easy to verify that each type environment $\mathcal{V}_\triangledown$ of previous definition agrees with the corresponding type theory $\nabla^\triangledown$ using the following proposition for $\mathcal{V}_{\mathrm{Scott}}$ and $\mathcal{V}_{\mathrm{CDZ}}$.

17.1.19. Proposition. (i) $\mathsf{PN} = \mathsf{N} \Rightarrow \mathsf{PN}$.
(ii) $\mathsf{PHN} = \Lambda \Rightarrow \mathsf{PHN}$.
(iii) $\mathsf{N} = \mathsf{PN} \Rightarrow \mathsf{N}$.

Proof. All cases are immediate but the inclusion $\mathsf{N} \subseteq \mathsf{PN} \Rightarrow \mathsf{N}$. Assume $M \in \mathsf{N}$ but there is $N \in \mathsf{PN}$ such that $MN \notin \mathsf{N}$. If N' is the normal form of N by Theorem 17.1.13 we get $N' \in \mathsf{NF}^+$. So by Definition 17.1.7 also $\lambda x.y(xN')$, where x is fresh, belongs to NF^+, and by the same theorem $\lambda x.y(xN) \in \mathsf{PN}$. But $\lambda x.y(xN)M \to a(MN) \notin \mathsf{N}$: we obtained a contradiction. ■

17.1.20. Definition. (Semantic Satisfability) Let $\rho : \mathsf{V} \to \Lambda$.
(i) $[\![M]\!]_\rho = M[\vec{x} := \vec{N}]$ where $\vec{x} = FV(M)$ and $\rho(\vec{x}) = \vec{N}$;
(ii) $\rho, \mathcal{V} \models M : A$ iff $[\![M]\!]_\rho \in [\![A]\!]_\mathcal{V}$;
(iii) $\rho, \mathcal{V} \models \Gamma$ iff $\rho, \mathcal{V} \models x : B$ for all $(x{:}B) \in \Gamma$;
(iv) $\Gamma \models^\triangledown M : A$ iff $\rho, \mathcal{V} \models \Gamma$ implies $\rho, \mathcal{V} \models M : A$ for all $\mathcal{V}$ which agree with $\nabla^\triangledown$, and all ρ. ■

17.1.21. Theorem. *(Soundness)* $\Gamma \vdash^\triangledown M : A$ *implies* $\Gamma \models^\triangledown M : A$.

Proof. By induction on derivations. The restriction to type environments which agree with $\nabla^{\triangledown}$ is essential for the soundness of rule $(\leq)$. ■

Proof of Theorem 17.1.5($\Rightarrow$).
Take $\rho_0(x) = x$. Notice that $\rho_0, \mathcal{V} \models \Gamma$ for all $\mathcal{V}$ and Γ such that if $(x{:}B)\in\Gamma$ then either $[\![B]\!]_{\mathcal{V}}$ is Λ or $[\![B]\!]_{\mathcal{V}}$ is S-type-stable for some S, since in both cases $[\![B]\!]_{\mathcal{V}}$ will contain all free variables.
(i). It easy to check using Proposition 17.1.16 that either $A \sim_{\mathrm{BCD}} \Omega$ and $[\![A]\!]_{\mathcal{V}^1_{\mathrm{BCD}}} = \Lambda$ or $A \not\sim_{\mathrm{BCD}} \Omega$ and $[\![A]\!]_{\mathcal{V}^1_{\mathrm{BCD}}}$ is HN-type-stable. So we have $\rho_0, \mathcal{V}^1_{\mathrm{BCD}} \models \Gamma$. From $\Gamma \vdash^{\mathrm{BCD}}_{\Omega} M : A$ by soundness we get $\Gamma \models^{\mathrm{BCD}} M : A$, i.e. $M = [\![M]\!]_{\rho_0}\in[\![A]\!]_{\mathcal{V}^1_{\mathrm{BCD}}} \subseteq \mathsf{HN}$, so we conclude $M\in\mathsf{HN}$.
(ii). For $\lambda\cap^{\mathrm{BCD}}_{\Omega}$ observe that by Proposition 17.1.16 $[\![A]\!]_{\mathcal{V}^2_{\mathrm{BCD}}}$ is N-type-stable whenever Ω does not occur in A. Therefore $\rho_0, \mathcal{V}^2_{\mathrm{BCD}} \models \Gamma$, since by hypothesis Ω does not occur in Γ. So as in case (i) we get by soundness $M\in\mathsf{N}$.
For $\lambda\cap^{\mathrm{CDZ}}_{\Omega}$ let Γ_ω be the basis all whose predicates are ω. By Definition 17.1.2 each free variable belongs to PN and therefore $\rho_0, \mathcal{V}_{\mathrm{CDZ}} \models \Gamma_\omega$. From $\Gamma_\omega \vdash^{\mathrm{CDZ}} M : \omega$ we get by soundness $M\in\mathsf{PN}$.
(iii). The proof is similar to that of case (ii) by observing that $[\![A]\!]_{\mathcal{V}_{\mathrm{CDV}}}$ is SN-type-stable for all A.
(iv) and (v). The proof is similar to that of case (ii) for $\lambda\cap^{\mathrm{CDZ}}_{\Omega}$ using respectively the type interpretations $\mathcal{V}_{\mathrm{Scott}}$ and $\mathcal{V}_{\mathrm{CDZ}}$. ■

Lasty we notice that the positive normal forms are exactly the persistent normal forms.

17.1.22. Proposition. $\mathsf{PNF} = \mathsf{NF}^+$.

Proof. The inclusion $\mathsf{NF}^+ \subseteq \mathsf{PNF}$ follows from the fact that the terms having all types in arrow type theories are persistently normaling, see Theorem 17.1.5(v)($\Rightarrow$) and positive normal forms can receive all types in arrow type theories, see Theorem 17.1.9. Lastly $\mathsf{PNF} \subseteq \mathsf{NF}^+$ by Theorem 17.1.13. ■

17.2. Realizability interpretation of types

In this section we discuss the set-theoretic semantics for intersection types, introduced in Barendregt et al. [1983], which generalizes the notion of type interpretation in λ-models given by Scott in Scott [1975] for simple types.

17.2.1. Definition. (i) Let $\mathcal{D} = \langle D, \cdot, [\![\ ]\!]\rangle$ be a λ-model. Given a subset $\Phi \subseteq D$ one can define the following intersection type structure

$$\mathcal{T}_{\mathcal{D},\Phi} = \langle \mathcal{P}(D), \cap, \rightsquigarrow_{\Phi}, \subseteq\rangle,$$

where $\cap$ and $\subseteq$ are as usual on $\mathcal{P}(D)$ and for $A, B \subseteq D$ one defines

$$A \rightsquigarrow_{\Phi} B = \{d\in\Phi \mid d \cdot A \subseteq B\},$$

where $d \cdot A = \{c \cdot a \mid a\in A\}$.

(ii) Given an intersection type structure $\mathcal{T} = \langle \mathcal{T}, \cap, \rightsquigarrow, \sqsubseteq \rangle$, a set of type atoms $\mathbb{A}$ and a map $\nu : \mathbb{A} \to \mathcal{T}$ one can define canonically

$$[\![\ ]\!]^{\mathcal{T}}_{\nu} : \mathbb{T}_{\cap}(\mathbb{A}) \to \mathcal{T}$$

as follows (writing $[\![A]\!]_{\nu}$ for $[\![A]\!]^{\mathcal{T}}_{\nu}$):

$$\begin{array}{rcl} [\![\alpha]\!]_{\nu} & = & \nu(\alpha), \text{ for } \alpha \in \mathbb{A}; \\ [\![A \cap B]\!]_{\nu} & = & [\![A]\!]_{\nu} \cap [\![B]\!]_{\nu}; \\ [\![A \to B]\!]_{\nu} & = & [\![A]\!]_{\nu} \rightsquigarrow [\![B]\!]_{\nu}. \end{array}$$

(iii) If $\mathcal{T} = \mathcal{T}_{\mathcal{D},\Phi}$ we write, $[\![A]\!]^{\mathcal{D},\Phi}_{\nu}$ for $[\![A]\!]^{\mathcal{T}}_{\nu}$.

(iv) A map $\nu : \mathbb{A} \to \mathcal{T}_{\mathcal{D},\Phi}$ agrees with the type theory ∇ if the map $[\![\ ]\!]^{\mathcal{D},\Phi}_{\nu} : \mathbb{T}_{\cap}(\mathbb{A}) \to \mathcal{T}_{\mathcal{D},\Phi}$ satfies

$$A \leq_{\nabla} B \ \Rightarrow\ [\![A]\!]^{\mathcal{D},\Phi}_{\nu} \subseteq [\![B]\!]^{\mathcal{D},\Phi}_{\nu}.$$

17.2.2. Definition (Semantic Satisfiability). Let $\mathcal{D} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}} \rangle$ be a λ-model and let $\Phi \subseteq D$. Let ∇ be a type theory over the atoms $\mathbb{A}$ and let $\nu : \mathbb{A} \to \mathcal{P}(D)$. Write $\mathcal{I} = (\mathcal{D}, \Phi)$. We define the following.

$$\begin{array}{lrcl} \text{(i)} & \mathcal{I}, \rho, \mathcal{V} \models M : A & \Leftrightarrow & [\![M]\!]_{\rho} \in [\![A]\!]^{\mathcal{D},\Phi}_{\mathcal{V}}; \\ & \mathcal{I}, \rho, \mathcal{V} \models \Gamma & \Leftrightarrow & \mathcal{I}, \rho, \mathcal{V} \models x : B, \text{ for all } (x{:}B) \in \Gamma; \\ & \Gamma \models^{\nabla} M : A & \Leftrightarrow & [\mathcal{I}, \rho, \mathcal{V} \models \Gamma \ \Rightarrow\ \mathcal{I}, \rho, \mathcal{V} \models M : A], \\ & & & \text{for all } \mathcal{I}, \rho \text{ and all } \nu \text{ agreeing with } \nabla. \end{array}$$

(ii) By modifying Φ many variations are possible. By making the maximal or minimal choice we get the following.

$$\begin{array}{rcl} \Gamma \models^{\nabla}_{s} M : A & \Leftrightarrow & [\mathcal{I}, \rho, \mathcal{V} \models \Gamma \ \Rightarrow\ \mathcal{I}, \rho, \mathcal{V} \models M : A], \\ & & \text{for all such } \mathcal{I}, \rho \text{ with } \Phi = D \\ & & \text{and } \nu \text{ agreeing with } \nabla. \\ \Gamma \models^{\nabla}_{F} M : A & \Leftrightarrow & [\mathcal{I}, \rho, \mathcal{V} \models \Gamma \ \Rightarrow\ \mathcal{I}, \rho, \mathcal{V} \models M : A], \\ & & \text{for all such } \mathcal{I}, \rho \text{ with } \Phi = \{[\![\lambda x.M]\!] \mid \lambda x.M \in \Lambda\} \\ & & \text{and } \nu \text{ agreeing with } \nabla. \end{array}$$

The main result in this section is that for every type theory ∇ one has

$$\Gamma \models^{\nabla} M : A \ \Leftrightarrow\ \Gamma \vdash^{\nabla} M : A.$$

This holds also for the two variants $\models^{\nabla}_{s}$ and $\models^{\nabla}_{F}$ of satisfaction.

17.2.3. Proposition (Soundness). *For all* ∇ *one has*

$$\Gamma \vdash^{\nabla} M : A \ \Rightarrow\ \Gamma \models^{\nabla} M : A.$$

Proof. Write $\vdash$ (respectively $\models$) for $\vdash^{\nabla}$ ($\models^{\nabla}$). By induction on the derivation of $\Gamma \vdash M : A$. Rules ($\to$E) and ($\cap$I) are trivially sound. As to the soundness of rule ($\to$I), assume $\Gamma, x{:}A \models M : B$ in order to show $\Gamma \models \lambda x.M : A{\to}B$. Assuming $\mathcal{I}, \rho, \nu \models \Gamma$ we have to show

$$\begin{array}{rcl}[\![\lambda x.M]\!]_\rho & \in & [\![A{\to}B]\!]_\nu \\ & = & [\![A]\!]_\nu \leadsto [\![B]\!]_\nu.\end{array}$$

Assume moreover $a{\in}[\![A]\!]_\nu$, then

$$[\![\lambda x.M]\!]_\rho(a) = [\![M]\!]_{\rho(x:=a)}{\in}[\![B]\!]_\nu,$$

as follows from $\Gamma, x{:}A \models M : B$, $a{\in}[\![A]\!]_\nu$ and therefore $\mathcal{I}, \rho_{(x:=a)}, \nu \models \Gamma, x{:}A$. Rule ($\leq$) is sound since we consider only consider maps ν that agree with $\nabla^\triangledown$. ■

As regards to completeness, first we observe that it is necessary that ∇ is a natural type theory.

17.2.4. Proposition. *(Completeness implies naturality) Suppose*

$$\Gamma \models^\triangledown M : A \;\Rightarrow\; \Gamma \vdash^\triangledown M : A,$$

for all Γ, M, A. *Then* $\nabla^\triangledown$ *is a* Ω*-(free) natural type theory.*

Proof. It is easy to verify that all type interpretations validate rule (η) and axiom ($\to$-$\cap$), and also axioms (Ω), (Ω-*lazy*) when $\Omega{\in}\mathbb{C}^\triangledown$, axiom ($\nu$) when $\nu{\in}\mathbb{C}^\triangledown$. For instance, as regards to axiom ($\to$-$\cap$), consider the $\triangledown$-basis $\Gamma = \{x{:}(A \to B) \cap (A \to C)\}$. From Definition 17.2.6 we get $\Gamma \models^\triangledown x : A{\to}B \cap C$. Hence, by hypothesis, we have $\Gamma \vdash^\triangledown x : A{\to}B \cap C$. From Theorem 15.1.16(**??**) it must hold $(A{\to}B) \cap (A{\to}C) \leq_\triangledown A{\to}B \cap C$, i.e. axiom ($\to$-$\cap$) must hold. ■

Now we shall discuss adequacy for each of the three semantics separately.

In particular we solve completely the characterization problem as far as the three canonical *set-theoretical* semantics for intersection-types: the inference semantics, the simple semantics Scott [1975] and the F-semantics Scott [1980a]. These are the semantics which arise by interpreting types as subsets of applicative structures, and by taking as interpretation of the *intersection constructor*, $\bigcap$, set-theoretic inclusion, and by taking the interpretation of the *arrow constructor*, $\to$, *à la* Scott, with respect to either *any possible functionality set*, or the *largest* one, or the *least* one.

More precisely the *simple semantics* of types associates to each arrow type $A \to B$ the set of elements in the domain which applied to an arbitrary element in the interpretation of A return an element in the interpretation of B.

As Scott has pointed out (Scott [1980a]), however, the key to a lambda model is the set of elements in the domain which are *canonical representatives* of functions, i.e. the elements which are meanings of terms starting with an initial abstraction. The F-semantics of types takes therefore as meaning of an arrow type only those elements which behave as expected with respect to the application *and* which are also canonical representatives of functions.

The inference semantics is the counterpart of the inference semantics for polymorphic types introduced in Mitchell [1988], generalized to arbitrary applicative structures. Here the interpretation of arrows is taken with respect to an arbitrary set which includes the canonical representatives of functions.

The results in this section strengthen and generalize significantly all earlier results in the literature, to our knowledge, in at least three respects. First of all the inference semantics had not been considered before. Secondly the characterizations are all given just in terms of simple closure conditions on the *preorder relation* , $\leq$, on the types, rather than on the typing judgments themselves, as had been done earlier Coppo et al. [1984], Dezani-Ciancaglini and Margaria [1986]. The task of checking the condition is made therefore considerably more tractable. Lastly we do not restrict attention just to λ-models, but to almost arbitrary applicative structures which admit an interpretation function (the λ-applicative structures introduced in Definition **??**). Thus we allow also for the treatment of models of restricted λ-calculi, and most notably models of Plotkin's call-by-value λ_v-calculus, and models of the λ-I-N-calculus of Honsell and Lenisa [1999]. Nevertheless the characterizations we give can be tailored just to the case of λ-models.

It is the set-theoretic semantics which gives the strongest evidence for the importance of Ω-(free) natural type theories. We show *soundness* for all type theories under suitable interpretations of types, *but* we show that only Ω-(free) natural type theories achieve *adequacy* for the *inference* semantics and hence *completeness* for the same semantics. Moreover, we characterize those intersection type theories which induce complete type assignment systems for the *simple* and the *F-semantics* over λ-applicative structures. As we will see these conditions apply also to the preorders which induce complete systems with respect to the three semantics over λ-models. Filter structures are essential for showing adequacy. In such structures, in fact, the set-theoretic interpretation of a type, as an appropriate subset, is in one-to-one correspondence with the *principal* filter generated by that type.

We give the general definition of type interpretation on suitable applicative structures, which allow for the interpretation of λ-terms, and have a distinguished set of functional objects. There are various choices for this set. Amongst these there is a *maximal one* and a *minimal* one. The former determines the *simple* semantics, the latter the *F-semantics.* Therefore type interpretations are parametrized on a suitable subset of the domain, the functionality set, and on a type environment.

17.2.5. Definition. (Type Interpretation Domain)

(i) A *type interpretation domain* is a quadruple $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\;]\!]^{\mathcal{D}}, \Phi \rangle$ such that:

(1) $\langle \mathcal{D}, \cdot, [\![\;]\!]^{\mathcal{D}} \rangle$ is a λ-applicative structure;

(2) Φ is a subset of $\mathcal{D}$, called the *functionality set*, and $[\![\lambda x.M]\!]^{\mathcal{D}}_{\rho} \in \Phi$ for all x, M, ρ.

(ii) A type interpretation domain $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\;]\!]^{\mathcal{D}}, \Phi \rangle$ is a *simple interpretation domain* if $\Phi = \mathcal{D}$;

(iii) A type interpretation domain $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}}, \Phi \rangle$ is an F-*interpretation domain* if $\Phi = \{d \in \mathcal{D} \mid d = [\![\lambda x.M]\!]^{\mathcal{D}}_{\rho}$ for some $x, M, \rho\}$. ∎

17.2.6. Definition (Type Interpretation). Let $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}}, \Phi \rangle$ be a type interpretation domain. The *type interpretation* induced by the type environment $\mathcal{V} : \mathbb{C}^{\nabla} \to \mathsf{P}(\mathcal{D})$ is defined by:

1. $[\![\Omega]\!]^{\mathcal{I}}_{\mathcal{V}} = \mathcal{D}$;
2. $[\![\nu]\!]^{\mathcal{I}}_{\mathcal{V}} = \Phi$;
3. $[\![\psi]\!]^{\mathcal{I}}_{\mathcal{V}} = \mathcal{V}(\psi)$ if $\psi \in \mathbb{C}^{\nabla}$ and $\psi \neq \Omega, \nu$;
4. $[\![A \to B]\!]^{\mathcal{I}}_{\mathcal{V}} = \{d \in \Phi \mid \forall e \in [\![A]\!]^{\mathcal{I}}_{\mathcal{V}}\ d \cdot e \in [\![B]\!]^{\mathcal{I}}_{\mathcal{V}}\}$;
5. $[\![A \cap B]\!]^{\mathcal{I}}_{\mathcal{V}} = [\![A]\!]^{\mathcal{I}}_{\mathcal{V}} \cap [\![B]\!]^{\mathcal{I}}_{\mathcal{V}}$. ∎

This definition is the counterpart for intersection-types of the *inference semantics* for polymorphic types of Mitchell [1988], generalized by allowing $\langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}} \rangle$ to be just a λ-applicative structure with an interpretation function for λ-terms instead of a λ-model.

Once we fix an λ-applicative structure $\langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}} \rangle$, the above definition depends on the choice of the functionality set Φ and of the type environment $\mathcal{V}$. The interpretation of the constants $\{\Omega, \nu\}$ takes into account the corresponding axioms of the type assignment systems.

Notice that in the definition of λ-applicative structure, we do not postulate, in general, that $[\![x]\!]^{\mathcal{D}}_{\rho} = \rho(x)$. Nevertheless the class of environments which have this property will be of particular significance (provided that they do not induce trivial interpretations, i.e. interpretations in which all terms are equated). Hence we put

17.2.7. Definition (Good environments). Let $\langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}} \rangle$ be a λ-applicative structure. The term environment $\rho : Var \to \mathcal{D}$ is *good* if for all $x \in Var$, we have $[\![x]\!]^{\mathcal{D}}_{\rho} = \rho(x)$ and moreover there exist two terms M, N such that $[\![M]\!]^{\mathcal{D}}_{\rho} \neq [\![N]\!]^{\mathcal{D}}_{\rho}$.

Clearly in the case of λ-structures all term environments are good and in the case of strict λ-structures all and only the $\perp$-free term environments are good. The environment ρ such that $\rho(x) = \perp$ for all variables x is not good since $[\![M]\!]^{s\mathcal{D}}_{\rho} = \perp$ for all terms M.

In discussing *sound* type assignment systems we consider only type interpretation domains and type environments which are good (the notion of goodness will depend on the current type theory and on which kind of semantics we are considering) and which agree with the inclusion relation between types in the following sense:

17.2.8. Definition. A type interpretation domain $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}}, \Phi \rangle$ and a type environment $\mathcal{V} : \mathbb{C}^{\nabla} \to \mathsf{P}(\mathcal{D})$

(i) are ∇-*good* if for all $A, B \in \mathbb{T}^{\nabla}$:

(1) for all *good* environments ρ and $d \in [\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$, $\rho[x := d]$ is *good*;

(2) for all *good* environments ρ, terms M and variables x,

$$\forall d{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}.\ [\![M]\!]^{\mathcal{D}}_{\rho[x:=d]}{\in}[\![B]\!]^{\mathcal{I}}_{\mathcal{V}} \ \Rightarrow\ \forall d{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}.[\![\lambda x.M]\!]^{\mathcal{D}}_{\rho}\cdot d{\in}[\![B]\!]^{\mathcal{I}}_{\mathcal{V}};$$

(ii) are $\bigtriangledown$-*F-good* if they are $\bigtriangledown$-good and moreover for all good environments ρ and $A{\in}\mathbb{T}^{\bigtriangledown}$

$$[\![x]\!]^{\mathcal{D}}_{\rho}{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}\cap\Phi \ \Rightarrow\ [\![\lambda y.xy]\!]^{\mathcal{D}}_{\rho}{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}};$$

(iii) *agree* with a type theory $\nabla^{\bigtriangledown}$ iff for all $A, B{\in}\mathbb{T}^{\bigtriangledown}$:

$$A\leq_{\bigtriangledown} B \ \Rightarrow\ [\![A]\!]^{\mathcal{I}}_{\mathcal{V}}\subseteq[\![B]\!]^{\mathcal{I}}_{\mathcal{V}}.$$

Notice that the condition for $\mathcal{I}$ and $\mathcal{V}$ being $\bigtriangledown$-good is satisfied whenever the λ-applicative structure is a (strict) natural λ-structure $\langle\mathcal{D},F^{\mathcal{D}},G^{\mathcal{D}}\rangle$ such that for all ρ, M and x,

$$[\![M]\!]^{(s)\mathcal{D}}_{\rho}=[\![(\lambda x.M)x]\!]^{(s)\mathcal{D}}_{\rho}$$

and moreover if $\langle\mathcal{D},F^{\mathcal{D}},G^{\mathcal{D}}\rangle$ is strict then $\perp\notin[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$ for all $A{\in}\mathbb{T}^{\bigtriangledown}$.

The case of strict λ-structure with not $\perp$-free environment is trivial.

Otherwise we have $[\![x]\!]^{(s)\mathcal{D}}_{\rho}=\rho(x)$. Therefore by choosing $\rho(x)=d$ we get

$$\begin{array}{rcl}[\![\lambda x.M]\!]^{(s)\mathcal{D}}_{\rho}\cdot d &=& [\![\lambda x.M]\!]^{(s)\mathcal{D}}_{\rho}\cdot[\![x]\!]^{(s)\mathcal{D}}_{\rho}\\ &=& [\![(\lambda x.M)x]\!]^{(s)\mathcal{D}}_{\rho}\\ &=& [\![M]\!]^{(s)\mathcal{D}}_{\rho}.\end{array}$$

This obviously forces condition (i2) of Definition 17.2.8.

As to $\bigtriangledown$-F-goodness of (strict) λ-structures, condition (ii) of Definition 17.2.8 follows whenever, for any $A{\in}\mathbb{T}^{\bigtriangledown}$, $[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$ is an upper closed set, and we consider the λ-structure as an F-interpretation domain by taking $\Phi=\{d{\in}\mathcal{D}\mid d=[\![\lambda x.M]\!]^{\mathcal{D}}_{\rho}$ for some $x,M,\rho\}$. In fact, in such a case, we have $\Phi\subseteq G_{\mathcal{D}}([\mathcal{D}\to\mathcal{D}])$, hence, if $[\![x]\!]^{\mathcal{D}}_{\rho}{\in}\Phi$, then there exists $f{\in}[\mathcal{D}\to\mathcal{D}]$ such that $G_{\mathcal{D}}(f)=[\![x]\!]^{\mathcal{D}}_{\rho}$. This implies $[\![x]\!]^{\mathcal{D}}_{\rho}\sqsubseteq[\![\lambda y.xy]\!]^{\mathcal{D}}_{\rho}$ by easy calculations, using $F_{\mathcal{D}}\circ G_{\mathcal{D}}\sqsupseteq Id_{[\mathcal{D}\to\mathcal{D}]}$. So, if $[\![x]\!]^{\mathcal{D}}_{\rho}{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$, then $[\![\lambda y.xy]\!]^{\mathcal{D}}_{\rho}{\in}[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$, since $[\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$ is upper closed.
Condition (ii) of Definition 17.2.8 holds also when $\mathcal{I}$ is an F-interpretation domain such that for all good ρ we get that $\rho(x){\in}\Phi$ implies $[\![x]\!]^{\mathcal{D}}_{\rho}=[\![\lambda y.xy]\!]^{\mathcal{D}}_{\rho}$.

Remark that the conditions (i) and (ii) of Definition 17.2.8 are true for all known models of (restricted) λ-calculus (Hindley and Longo [1980], Plotkin [1975], Egidi et al. [1992], Honsell and Lenisa [1999]).

One can easily see that the following holds:

17.2.9. PROPOSITION.

1. *All type interpretation domains and all type environments agree with AO and with EHR.*

2. *All simple interpretation domains and all type environments agree with BCD.* ∎

We now introduce formally the three semantics. The definitions are natural given how we have argued informally so far, but for the restriction (in the definition of $\models^{\triangledown}$) to those type interpretation domains and type environments which are $\triangledown$-good ($\triangledown$-F-good in the case of F-semantics) and which agree with $\nabla^{\triangledown}$.

17.2.10. Definition (Semantic Satisfiability). Let $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^{\mathcal{D}}, \Phi\rangle$ be a type interpretation domain.

(i) $\mathcal{I}, \rho, \mathcal{V} \models M : A$ iff $[\![M]\!]^{\mathcal{D}}_{\rho} \in [\![A]\!]^{\mathcal{I}}_{\mathcal{V}}$;

(ii) $\mathcal{I}, \rho, \mathcal{V} \models \Gamma$ iff $\mathcal{I}, \rho, \mathcal{V} \models x : B$ for all $(x{:}B)\in\Gamma$;

(iii) $\Gamma \models^{\triangledown}_{i} M : A$ iff $\mathcal{I}, \rho, \mathcal{V} \models \Gamma$ implies $\mathcal{I}, \rho, \mathcal{V} \models M : A$ for all $\triangledown$-good type interpretation domains $\mathcal{I}$ and type environments $\mathcal{V}$ which moreover agree with $\nabla^{\triangledown}$, and for all good term environments ρ;

(iv) $\Gamma \models^{\triangledown}_{s} M : A$ iff $\mathcal{I}, \rho, \mathcal{V} \models \Gamma$ implies $\mathcal{I}, \rho, \mathcal{V} \models M : A$ for all $\triangledown$-good simple interpretations domains $\mathcal{I}$ and type environments $\mathcal{V}$ which moreover agree with $\nabla^{\triangledown}$, and for all good term environments ρ;

(v) $\Gamma \models^{\triangledown}_{F} M : A$ iff $\mathcal{I}, \rho, \mathcal{V} \models \Gamma$ implies $\mathcal{I}, \rho, \mathcal{V} \models M : A$ for all $\triangledown$-F-good F-interpretations domains $\mathcal{I}$ and type environments $\mathcal{V}$ which moreover agree with $\nabla^{\triangledown}$, and for all good term environments ρ.

Exercise 16.6.28 may be useful for understanding some differences between the three notions of semantic satisfiability.

In view of the above definition, we can say that the *inference semantics* is given by $\models^{\triangledown}_{i}$, the *simple semantics* by $\models^{\triangledown}_{s}$, and the *F-semantics* by $\models^{\triangledown}_{F}$. The following proposition is immediate:

17.2.11. Proposition. *If* $\Gamma \models^{\triangledown}_{i} M : A$ *then we have both* $\Gamma \models^{\triangledown}_{s} M : A$ *and* $\Gamma \models^{\triangledown}_{F} M : A$.

Notation. We shall denote with $\models^{\triangledown}$ any of the three $\models^{\triangledown}_{i}$, $\models^{\triangledown}_{s}$, and $\models^{\triangledown}_{F}$.

Derivability in the type system implies semantic satisfiability, as shown in the next theorem.

17.2.12. Theorem (Soundness). $\Gamma \vdash^{\triangledown} M : A$ *implies* $\Gamma \models^{\triangledown} M : A$. ∎

Proof. By induction on the derivation of $\Gamma \vdash^{\triangledown} M : A$.
Rules ($\to$E) and ($\cap$I) are sound by the definition of type interpretation (Definition 17.2.6).
The soundness of rule ($\to$I) follows from the restriction to $\triangledown$-good type interpretation domains and type environments (Definition 17.2.8(i)) and from the definition of functionality set (Definition 17.2.5(i2)).
Rule ($\leq_{\triangledown}$) is sound since we consider only type interpretation domains and type environments which agree with $\nabla^{\triangledown}$ (Definition 17.2.8(iii)). ∎

First we consider the inference semantics. Our goal is to show that all Ω-(free) natural type theories are adequate for the inference semantics.
For this proof, we focus on the applicative structure induced by the filter structure $\langle \mathcal{F}^{\triangledown}, F^{\triangledown}_{\diamond\diamond}, G^{\triangledown}\rangle$, and we put:

17.2.13. DEFINITION. Let $\nabla^\triangledown$ be a Ω-(free) natural type theory. Let:

(i) $\Phi^\triangledown$ be the functionality set defined by

$$\Phi^\triangledown = \begin{cases} \{X\in\mathcal{F}^\triangledown \mid \Omega\to\Omega\in X\} & \text{if } \Omega\in\mathbb{C}^\triangledown; \\ \{X\in\mathcal{F}^\triangledown \mid \nu\in X\} & \text{if } \nu\in\mathbb{C}^\triangledown; \\ \mathcal{F}^\triangledown & \text{otherwise.} \end{cases}$$

(ii) $\mathcal{I}^\triangledown$ be the type interpretation domain $\langle\mathcal{F}^\triangledown, \cdot, [\![\ ]\!]^\triangledown, \Phi^\triangledown\rangle$.

(iii) $\mathcal{V}^\triangledown : \mathbb{C}^\triangledown \to \mathsf{P}(\mathcal{F}^\triangledown)$ be the type environment defined by

$$\mathcal{V}^\triangledown(\psi) = \{X\in\mathcal{F}^\triangledown \mid \psi\in X\}.$$

(iv) $[\![\ ]\!]^\triangledown : \mathbb{T}^\triangledown \to \mathsf{P}(\mathcal{F}^\triangledown)$ be the mapping $[\![\ ]\!]^{\mathcal{I}^\triangledown}_{\mathcal{V}^\triangledown}$. ∎

17.2.14. REMARK. The symbol $[\![\ ]\!]^\triangledown$ is overloaded, since it denotes both term interpretation (according to Definition **??**) and type interpretation (according to Definition 17.2.13) for the domain $\mathcal{I}^\triangledown$. This is however harmless, since it is always clear from the context which of the two interpretations we are using.

Notice that

- when $\Omega\in\mathbb{C}^\triangledown$ we get $[\![\Omega]\!]^\triangledown = \mathcal{F}^\triangledown = \{X\in\mathcal{F}^\triangledown \mid \Omega\in X\} = \mathcal{V}^\triangledown(\Omega)$ and $[\![\Omega\to\Omega]\!]^\triangledown = \{X\in\Phi^\triangledown \mid \Omega\to\Omega\in X\} = \Phi^\triangledown$;
- when $\nu\in\mathbb{C}^\triangledown$ we get $[\![\nu]\!]^\triangledown = \Phi^\triangledown = \{X\in\Phi^\triangledown \mid \nu\in X\} = \mathcal{V}^\triangledown(\nu)$.

The mapping $[\![\ ]\!]^\triangledown : \mathbb{T}^\triangledown \to \mathsf{P}(\mathcal{F}^\triangledown)$ has the property of associating to each type A the set of filters which contain A (thus preserving the property which defines $\mathcal{V}^\triangledown$ in the basic case of type constants).

17.2.15. LEMMA. *Let $\nabla^\triangledown$ be a Ω-(free) natural type theory then*

$$[\![A]\!]^\triangledown = \{X\in\mathcal{F}^\triangledown \mid A\in X\}.$$

PROOF. By induction on A. The only interesting case is when A is an arrow type. Remark that if $X\in\mathcal{F}^\triangledown$ but $X\notin\Phi^\triangledown$ then all types in X are intersections of constant types. In fact if X contains an arrow type, then it contains also $\Omega\to\Omega$ when $\Omega\in\mathbb{C}^\triangledown$ (by rule (Ω-*lazy*)), or ν when $\nu\in\mathbb{C}^\triangledown$ (by rule (ν)), so X belongs to $\Phi^\triangledown$.
If $A\equiv B\to C$ we have

$$\begin{array}{rcll} [\![B\to C]\!]^\triangledown & = & \{X\in\Phi^\triangledown \mid \forall Y\in[\![B]\!]^\triangledown.\, X\cdot Y\in[\![C]\!]^\triangledown\} & \text{by definition} \\ & = & \{X\in\Phi^\triangledown \mid \forall Y.\, B\in Y \Rightarrow C\in X\cdot Y\} & \text{by induction} \\ & = & \{X\in\Phi^\triangledown \mid C\in X\cdot\uparrow B\} & \text{by monotonicity} \\ & = & \{X\in\Phi^\triangledown \mid B\to C\in X\} & \text{by Proposition ??} \\ & & & \text{since } \Omega\to\Omega\in X \text{ when } \Omega\in\mathbb{C}^\triangledown \\ & & & \text{by the definition of } \Phi^\triangledown \\ & = & \{X\in\mathcal{F}^\triangledown \mid B\to C\in X\} & \text{by above. } \blacksquare \end{array}$$

17.2.16. LEMMA. *Let $\nabla^\triangledown$ be a Ω-(free) natural type theory. Then $\mathcal{I}^\triangledown$, $\mathcal{V}^\triangledown$ are $\triangledown$-good and they agree with $\nabla^\triangledown$.*

PROOF. $\mathcal{I}^\nabla$, $\mathcal{V}^\nabla$ satisfy condition (i1) of Definition 17.2.8 since $\emptyset \notin [\![A]\!]^\nabla$ for all $A \in \mathbb{T}^\nabla$ by Lemma 17.2.15.
For condition (i2) of Definition 17.2.8 let $X \in [\![A]\!]^\nabla$ be such that $[\![M]\!]^\nabla_{\rho[x:=X]} \in [\![B]\!]^\nabla$. Then, by Lemma 17.2.15, $B \in [\![M]\!]^\nabla_{\rho[x:=X]}$, hence $B \in f(X)$, where we have put $f = \boldsymbol{\lambda} d.[\![M]\!]^\nabla_{\rho[x:=d]}$. Since, by Theorem **??** (and Definition **??**), $f \sqsubseteq F^\nabla_{\infty}(G^\nabla(f))$, we have $B \in F^\nabla_{\infty}(G^\nabla(f))(X)$. We are done since $F^\nabla_{\infty}(G^\nabla(f))(X) = [\![\lambda x.M]\!]^\nabla_\rho \cdot X$.

Lastly notice that as an immediate consequence of the Lemma 17.2.15 we get

$$A \leq_\nabla B \Leftrightarrow \forall X \in \mathcal{F}^\nabla.[A \in X \Rightarrow B \in X] \Leftrightarrow [\![A]\!]^\nabla \subseteq [\![B]\!]^\nabla,$$

and therefore $\mathcal{I}^\nabla, \mathcal{V}^\nabla$ agree with ∇^∇. ■

Finally we can prove the desired adequacy result.

17.2.17. THEOREM. *(Naturality implies adequacy) Let ∇^∇ be a Ω-(free) natural type theory, then $\Gamma \models^\nabla_i M : A$ implies $\Gamma \vdash^\nabla M : A$.*

PROOF. We consider the type interpretation domain $\mathcal{I}^\nabla$. Let ρ_Γ be the term environment defined by $\rho_\Gamma(x) = \{A \in \mathbb{T}^\nabla \mid \Gamma \vdash^\nabla x : A\}$. It is easy to verify that $\mathcal{I}^\nabla, \rho_\Gamma, \mathcal{V}^\nabla \models \Gamma$ and that for all $\Gamma' \models \rho_\Gamma$ we have $\Gamma' \vdash^\nabla M : A \Rightarrow \Gamma \vdash^\nabla M : A$. Hence we have:

$$\begin{array}{rcll} \Gamma \models^\nabla_i M : A & \Rightarrow & [\![M]\!]^\nabla_{\rho_\Gamma} \in [\![A]\!]^\nabla & \text{by Lemma 17.2.16 since } \mathcal{I}^\nabla, \rho_\Gamma, \mathcal{V}^\nabla \models \Gamma \\ & \Rightarrow & A \in [\![M]\!]^\nabla_{\rho_\Gamma} & \text{by Lemma 17.2.15} \\ & \Rightarrow & \Gamma \vdash^\nabla M : A & \text{by Theorem 16.4.4 and} \\ & & & \text{the above property. } \blacksquare \end{array}$$

Hence the only type theories which turn out to be complete with respect to the inference semantics are the Ω-(free) natural type theories. By Theorem **??** the theories ∇^{AO}, ∇^{BCD}, ∇^{Scott}, ∇^{Park}, ∇^{CDZ}, ∇^{HR} induce filter structures which are λ-models, the theory ∇^{EHR} induces a model for the λ_v-calculus (Egidi et al. [1992]), and the theories ∇^{CDV}, ∇^{HL} induce models for the λ-I-N-calculus (Honsell and Lenisa [1999]). Hence we have that Ω-natural theories, which induce λ-models, are complete also for the class of λ-models, and Ω-free natural theories, which induce models of the other two restricted λ-calculi, are complete for the corresponding classes of models.

Now we characterize those theories which are complete with respect to the simple semantics.

17.2.18. THEOREM. *(Adequacy for the simple semantics) $\Gamma \models^\nabla_s M : A$ implies $\Gamma \vdash^\nabla M : A$ iff ∇^∇ is a Ω-free natural type theory such that $\nu \notin \mathbb{C}^\nabla$ or a Ω-natural type theory which validates axiom (Ω-η).*

PROOF. ($\Rightarrow$) From Proposition 17.2.4 it follows that ∇^∇ is Ω-(free) natural. If $\nu \in \mathbb{C}^\nabla$ we have, for any type interpretation domain $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\ ]\!]^\mathcal{D}, \Phi \rangle$ and $\mathcal{V}$ type environment: $[\![\nu]\!]^\mathcal{I}_\mathcal{V} = \Phi = \mathcal{D}$, hence $\models^\nabla_s x : \nu$. But it never holds $\vdash^\nabla_\nu x : \nu$

by Theorem 15.1.16(**??**), hence simple adequacy fails if $\nu \in \mathbb{C}^{\nabla}$.
It is easy to check that if $\Omega \not\sim_{\nabla} \Omega \to \Omega$ then simple adequacy fails for $\lambda\cap^{\nabla}$. We have $\models_s^{\nabla} x : \Omega \to \Omega$ since $[\![x]\!]_{\rho}^{\mathcal{D}} \cdot d \in \mathcal{D}$ for all $\mathcal{D}, d \in \mathcal{D}$ and $\rho : \mathsf{Env}_{\mathcal{D}} \to \mathcal{D}$. By Theorem 15.1.16(**??**) we can deduce $\vdash_{\Omega}^{\nabla} x : \Omega \to \Omega$ only if $\Omega \sim_{\nabla} \Omega \to \Omega$.
This proves $(\Rightarrow)$.
$(\Leftarrow)$. To prove that $\Gamma \models_s^{\nabla} M : A$ implies $\Gamma \vdash^{\nabla} M : A$ under the given conditions we use the simple type interpretation domain $\langle \mathcal{F}^{\nabla}, \cdot, [\![\ ]\!]^{\nabla}, \mathcal{F}^{\nabla} \rangle$, which is just $\mathcal{I}^{\nabla}$ as defined in Definition 17.2.13, since $\Phi^{\nabla} = \mathcal{F}^{\nabla}$, being either $\nu \notin \mathbb{C}^{\nabla}$ or $\Omega \sim_{\nabla} \Omega \to \Omega$. By Lemma 17.2.15 it follows $[\![A]\!]^{\nabla} = \{X \in \mathcal{F}^{\nabla} \mid A \in X\}$. So we have that $\Gamma \models_s^{\nabla} M : A$ implies $A \in [\![M]\!]_{\rho_\Gamma}^{\nabla}$ and we conclude $\Gamma \vdash^{\nabla} M : A$ as in the last step of the proof of Theorem 17.2.17. ■

Among the type theories of Figure 17.3, those adequate for the simple semantics are ∇^{CDV}, ∇^{HL}, ∇^{BCD}, ∇^{Scott}, ∇^{Park}, ∇^{CDZ}, and ∇^{HR}. Non adequate type theories are all those inducing computationally adequate models for the lazy λ-calculus or for the call-by-value λ-calculus, e.g. ∇^{AO} or ∇^{EHR}. The same argument used for the inference semantics allows to show that the Ω-natural theories mentioned in Theorem 17.2.18, which induce λ-models, are precisely those which are complete also for the class of λ-models, and the Ω-free natural theories, which induce λ_v-models and λ-I-N-models, are complete for the corresponding classes of models. The completeness for the simple semantics of $\lambda\cap_{\Omega}^{\nabla}$ whenever ∇^{∇} validates BCD and $\langle \mathcal{F}^{\nabla}, \cdot, [\![\ ]\!]^{\nabla} \rangle$ is a λ-model was proved in Coppo et al. [1984] using filter models and in Coppo et al. [1987] using the term model of β-equality.

Finally we turn to the F-semantics. The following definition singles out the type theories which are adequate for the F-semantics as proved in Theorem 17.2.27.

17.2.19. Definition. A type theory ∇^{∇} is an F-*type theory* iff

1. either ∇^{∇} is a Ω-(free) natural type theory such that $\nu \notin \mathbb{C}^{\nabla}$ and for all $\psi \in \mathbb{C}^{\nabla}$, $A, B \in \mathbb{T}^{\nabla}$, $\psi \cap (A \to B) \sim_{\nabla} \bigcap_{i \in I}(A_i \to B_i)$, for some $I, A_i, B_i \in \mathbb{T}^{\nabla}$;
2. or ∇^{∇} is a Ω-free natural type theory such that $\nu \in \mathbb{C}^{\nabla}$ and for all $\psi \in \mathbb{C}^{\nabla}$ either $\nu \leq_{\nabla} \psi$ or $\psi \cap \nu \sim_{\nabla} \bigcap_{i \in I}(A_i \to B_i)$, for some $I, A_i, B_i \in \mathbb{T}^{\nabla}$. ■

For example ∇^{HL}, ∇^{EHR}, and ∇^{AO} are F-type theories.

Next lemma shows that all types of a F-type theory satisfy the conditions of previous definition.

17.2.20. Lemma. *Let ∇^{∇} be a F-type theory. Then*

(i) *if $\nu \notin \mathbb{C}^{\nabla}$, then for all $A, B, C \in \mathbb{T}^{\nabla}$, $C \cap (A \to B) \sim_{\nabla} \bigcap_{i \in I}(A_i \to B_i)$, for some $I, A_i, B_i \in \mathbb{T}^{\nabla}$;*

(ii) *if $\nu \in \mathbb{C}^{\nabla}$, then for all $C \in \mathbb{T}^{\nabla}$ either $\nu \leq_{\nabla} C$ or $C \cap \nu \sim_{\nabla} \bigcap_{i \in I}(A_i \to B_i)$, for some $I, A_i, B_i \in \mathbb{T}^{\nabla}$;*

(iii) *for all* $A,B,C\in\mathbb{T}^{\triangledown}$, $C\cap(A\to B)\sim_{\triangledown}\bigcap_{i\in I}(A_i\to B_i)$, *for some* I, $A_i,B_i\in\mathbb{T}^{\triangledown}$.

PROOF. We just prove the more difficult case, namely (ii). We reason by induction on the structure of C. If $C\in\mathbb{C}^{\triangledown}$ the thesis is trivial. If $C\equiv D\to E$, then $C\leq_{\triangledown}\nu$, hence $C\cap\nu\sim_{\triangledown}D\to E$. If $C\equiv D\cap E$ and $\nu\leq_{\triangledown}C$ then the thesis is immediate. Otherwise we cannot have both $\nu\leq_{\triangledown}D$ and $\nu\leq_{\triangledown}E$. Let us suppose $\nu\not\leq_{\triangledown}D$. Then, by induction, it follows $D\cap\nu\sim_{\triangledown}\bigcap_{i\in I}(A_i\to B_i)$, for suitable I and $A_i,B_i\in\mathbb{T}^{\triangledown}$. Now, if $\nu\leq_{\triangledown}E$, we get $C\cap\nu\sim_{\triangledown}D\cap\nu$ and we are done by above. If $\nu\not\leq_{\triangledown}E$, then, by induction, it follows $E\cap\nu\sim_{\triangledown}\bigcap_{j\in J}(A'_j\to B'_j)$ for suitable J and $A'_j,B'_j\in\mathbb{T}^{\triangledown}$. Therefore $C\cap\nu\sim_{\triangledown}(\bigcap_{i\in I}(A_i\to B_i))\cap(\bigcap_{j\in J}(A'_j\to B'_j))$. ■

To discuss F-semantics it is useful to characterize the subset of types which are functional.

17.2.21. DEFINITION. We define the predicate fun on $\mathbb{T}^{\triangledown}$ by induction on the structure of types:

1. $\mathsf{fun}(\psi)=\psi\sim_{\triangledown}\nu$ or $\psi\sim_{\triangledown}\bigcap_{i\in I}(A_i\to B_i)$ for some $I,A_i,B_i\in\mathbb{T}^{\triangledown}$;
2. $\mathsf{fun}(A\to B)=\mathsf{true}$;
3. $\mathsf{fun}(A\cap B)=\mathsf{fun}(A)$ or $\mathsf{fun}(B)$. ■

The following proposition gives an alternative characterization of functional types for F-type theories.

17.2.22. PROPOSITION. *If* $\nabla^{\triangledown}$ *is an F-type theory then* $\mathsf{fun}(A)$ *iff either* $A\sim_{\triangledown}\nu$ *or* $A\sim_{\triangledown}\bigcap_{i\in I}(A_i\to B_i)$ *for some* $I,A_i,B_i\in\mathbb{T}^{\triangledown}$. ■

PROOF. ($\Leftarrow$) is trivial.

($\Rightarrow$) We reason by induction on the structure of A. If $A\in\mathbb{C}^{\triangledown}$, or $A\equiv B\to C$, or $A\sim_{\triangledown}\nu$, the thesis follows by definition of $\mathsf{fun}(A)$. Otherwise we have $A\equiv B\cap C$ and either $\mathsf{fun}(B)$ or $\mathsf{fun}(C)$. We assume $\mathsf{fun}(B)$, the case $\mathsf{fun}(C)$ being similar. By induction $B\sim_{\triangledown}\nu$ or $B\sim_{\triangledown}\bigcap_{i\in I}(A_i\to B_i)$ for some I and $A_i,B_i\in\mathbb{T}^{\triangledown}$. In the first case either $\nu\leq_{\triangledown}C$ and $A\sim_{\triangledown}C\cap\nu\sim_{\triangledown}\nu$ or $A\sim_{\triangledown}C\cap\nu\sim_{\triangledown}\bigcap_{i\in J}(A'_j\to B'_j)$, for some J and $A'_j,B'_j\in\mathbb{T}^{\triangledown}$ by Lemma 17.2.20(ii). In the second case $A\sim_{\triangledown}C\cap\bigcap_{i\in I}(A_i\to B_i)$. By choosing an arbitrary $i\in I$ we get $C\cap(A_i\to B_i)\sim_{\triangledown}\bigcap_{i\in J}(A'_j\to B'_j)$, for some J and $A'_j,B'_j\in\mathbb{T}^{\triangledown}$ by Lemma 17.2.20(iii). So we conclude $A\sim_{\triangledown}(\bigcap_{i\in I}(A_i\to B_i))\cap(\bigcap_{i\in J}(A'_j\to B'_j))$.

To prove adequacy we will again use the filter structure $\langle\mathcal{F}^{\triangledown},F^{\triangledown}_{\infty},G^{\triangledown}\rangle$, but with a different type interpretation domain and type environment.

17.2.23. DEFINITION. Let $\nabla^{\triangledown}$ be an F-type theory. Let:

1. $\Phi^{\triangledown}_{F}$ be the functionality set defined by

$$\Phi^{\triangledown}_{F}=\{X\in\mathcal{F}^{\triangledown}\mid X=[\![\lambda x.M]\!]^{\triangledown}_{\rho}\text{ for some }\ x,M,\rho\}.$$

2. $\mathcal{I}_F^{\triangledown}$ be the type interpretation domain $\langle \mathcal{F}^{\triangledown}, \cdot, [\![\ ]\!]^{\triangledown}, \Phi_F^{\triangledown} \rangle$.

3. $\mathcal{V}_F^{\triangledown} : \mathbb{C}^{\triangledown} \to \mathsf{P}(\mathcal{F}^{\triangledown})$ be the type environment defined by

$$\mathcal{V}_F^{\triangledown}(\psi) = \{X \in \mathcal{F}^{\triangledown} \mid \psi \in X\}.$$

4. $[\![\ ]\!]_F^{\triangledown} : \mathbb{T}^{\triangledown} \to \mathsf{P}(\mathcal{F}^{\triangledown})$ be the mapping $[\![\ ]\!]_{\mathcal{V}_F^{\triangledown}}^{\mathcal{I}_F^{\triangledown}}$. ■

When restricting to F-type theories, all filters which contain a functional type belong to the functionality set.

17.2.24. LEMMA. *Let $\nabla^{\triangledown}$ be an F-type theory and $X \in \mathcal{F}^{\triangledown}$. Then $A \in X$ and $\mathsf{fun}(A)$ imply $X \in \Phi_F^{\triangledown}$.*

PROOF. We show that under the given conditions

$$X = [\![\lambda y.xy]\!]_{\rho_0}^{\triangledown} \text{ where } \rho_0(x) = X.$$

Proof of $X \subseteq [\![\lambda y.xy]\!]_{\rho_0}^{\triangledown}$. Take an arbitrary $B \in X$. Notice that if $\nu \in \mathbb{C}^{\triangledown}$ then $\mathsf{fun}(A)$ implies $A \leq_{\triangledown} \nu$ by Proposition 17.2.22. Moreover $\mathsf{fun}(A)$ implies $\mathsf{fun}(A \cap B)$ by Definition 17.2.21. Then either $A \cap B \sim_{\triangledown} \nu$ or $A \cap B \sim_{\triangledown} \bigcap_{i \in I}(C_i \to D_i)$ for some $I, C_i, D_i \in \mathbb{T}^{\triangledown}$ again by Proposition 17.2.22. In the first case we get $\nu \leq_{\triangledown} B$ and then $\vdash_{\nu}^{\triangledown} \lambda y.xy : B$ by axiom (Ax-ν) and rule ($\leq_{\triangledown}$). In the second case we can derive $\{x{:}\bigcap_{i \in I}(C_i \to D_i)\} \vdash^{\triangledown} \lambda y.xy : \bigcap_{i \in I}(C_i \to D_i)$ using axiom (Ax) and rules ($\leq_{\triangledown}$), ($\to$E), ($\to$I), and ($\cap$I). This implies $\{x{:}A \cap B\} \vdash^{\triangledown} \lambda y.xy : B$ by rules ($\leq_{\triangledown}$) and ($\leq_{\triangledown}$ L). In both cases we conclude $B \in [\![\lambda y.xy]\!]_{\rho_0}^{\triangledown}$ since $\emptyset \models \rho_0$ and $\{x{:}A \cap B\} \models \rho_0$.
Proof of $[\![\lambda y.xy]\!]_{\rho_0}^{\triangledown} \subseteq X$. By Theorem 16.4.4, $B \in [\![\lambda y.xy]\!]_{\rho_0}^{\triangledown}$ implies $\{x{:}C\} \vdash^{\triangledown} \lambda y.xy : B$ for some $C \in X$. If $\Omega \in \mathbb{C}^{\triangledown}$ and $B \sim_{\triangledown} \Omega$ then $B \in X$ for all X. If $\nu \in \mathbb{C}^{\triangledown}$ and $\nu \leq_{\triangledown} B$ then $B \in X$ since $A \leq_{\triangledown} \nu$ by Proposition 17.2.22. Otherwise we get $\{x{:}C, y{:}D_i\} \vdash^{\triangledown} xy : E_i$ for some $I, D_i, E_i \in \mathbb{T}^{\triangledown}$ such that $\bigcap_{i \in I}(D_i \to E_i) \leq_{\triangledown} B$ by Theorem 15.1.2(**??**). This implies $\{x{:}C, y{:}D_i\} \vdash^{\triangledown} x : F_i \to E_i$, $\{x{:}C, y{:}D_i\} \vdash^{\triangledown} y : F_i$ for some $F_i \in \mathbb{T}^{\triangledown}$ by Theorem 15.1.16(**??**). Using Theorem 15.1.16(**??**) we have $C \leq_{\triangledown} F_i \to E_i$ and $D_i \leq_{\triangledown} F_i$ for all $i \in I$, so we get $C \leq_{\triangledown} D_i \to E_i$ by rule (η), and we can conclude $C \leq_{\triangledown} B$, i.e. $B \in X$. ■

17.2.25. LEMMA. *Let $\nabla^{\triangledown}$ be a F-type theory then*

$$[\![A]\!]_F^{\triangledown} \quad = \quad \{X \in \mathcal{F}^{\triangledown} \mid A \in X\}.$$

PROOF. The proof by induction on A is similar to that of Lemma 17.2.15. All cases are trivial but ν and arrow types.
If $A \equiv \nu$ let X be any filter in $\Phi_F^{\triangledown}$, that is $X = [\![\lambda x.M]\!]_\rho^{\triangledown}$ for some x, M, ρ. Then, by Theorem 16.4.4, $X = \{B \in \mathbb{T}^{\triangledown} \mid \exists \Gamma \models \rho.\ \Gamma \vdash_{\nu}^{\triangledown} \lambda x.M : B\}$. Since $\vdash_{\nu}^{\triangledown} \lambda x.M : \nu$, we have $\nu \in X$. Vice versa, if $\nu \in X$, then by Definition 17.2.21 $\mathsf{fun}(\nu)$, and so by Lemma 17.2.24, $X \in \Phi_F^{\triangledown}$. We have proved, when $\nu \in \mathbb{C}^{\triangledown}$, that

$$X \in \Phi_F^{\triangledown} \Leftrightarrow \nu \in X.$$

Hence $[\![\nu]\!]_F^\triangledown = \Phi_F^\triangledown = \{X{\in}\mathcal{F}^\triangledown \mid \nu{\in}X\}$.
If $A \equiv B \to C$ we have

$$\begin{array}{rcll}
[\![B{\to}C]\!]_F^\triangledown & = & \{X{\in}\Phi_F^\triangledown \mid C{\in}X\cdot\uparrow B\} & \text{as in the proof of Lemma 17.2.15}\\
& = & \{X{\in}\Phi_F^\triangledown \mid B{\to}C{\in}X\} & \text{by Proposition ?? since,}\\
& & & \text{when } \Omega{\in}\mathbb{C}^\triangledown\text{, Theorem 16.4.4}\\
& & & \text{and } X = [\![\lambda x.M]\!]_\rho^\triangledown \text{ imply } \Omega \to \Omega{\in}X\\
& = & \{X{\in}\mathcal{F}^\triangledown \mid B{\to}C{\in}X\} & \text{by Lemma 17.2.24 since } \mathsf{fun}(B \to C). \blacksquare
\end{array}$$

17.2.26. LEMMA. *Let $\nabla^\triangledown$ be an F-type theory. Then $\mathcal{I}_F^\triangledown$, $\mathcal{V}_F^\triangledown$ are $\triangledown$-F-good and they agree with $\nabla^\triangledown$.*

PROOF. We can mimick the proof of Lemma 17.2.16, using Lemma 17.2.25 instead of Lemma 17.2.15, for all points of Definition 17.2.8 but (ii). So we are left to prove that this last condition holds. The key observation is that by Lemma 17.2.25 and Theorem 16.4.4

$$(*)\quad [\![M]\!]_\rho^\triangledown{\in}[\![A]\!]_F^\triangledown \;\Leftrightarrow\; \Gamma \vdash^\triangledown M : A \text{ for some } \triangledown\text{-basis } \Gamma \text{ such that } \Gamma \models \rho.$$

Let $[\![x]\!]_\rho^\triangledown{\in}[\![A]\!]_F^\triangledown{\cap}\Phi_F^\triangledown$. Then $[\![x]\!]_\rho^\triangledown = [\![\lambda z.M]\!]_{\rho'}^\triangledown$ for some z, M, ρ'. By $(*)$ there exists a $\triangledown$-basis Γ' such that $\Gamma' \models \rho'$ and $\Gamma' \vdash^\triangledown \lambda z.M : A$. By Theorem 15.1.2(**??**) there exist I, A_i, B_i, such that $\Gamma' \vdash \lambda z.M : \bigcap_{i\in I}(A_i \to B_i)$ and $\bigcap_{i\in I}(A_i \to B_i) \leq_\triangledown A$. Hence by $(*)$ $[\![\lambda z.M]\!]_{\rho'}^\triangledown{\in}[\![\bigcap_{i\in I}(A_i \to B_i)]\!]_F^\triangledown$. Since $[\![x]\!]_\rho^\triangledown = [\![\lambda z.M]\!]_{\rho'}^\triangledown$ by $(*)$ there exists a $\triangledown$-basis $\Gamma \models \rho$, such that $\Gamma \vdash^\triangledown x : \bigcap_{i\in I}(A_i \to B_i)$, hence $\Gamma \vdash^\triangledown \lambda y.xy : \bigcap_{i\in I}(A_i \to B_i)$, by applying rules $(\leq_\triangledown)$, $(\to$E$)$, $(\to$I$)$ and $(\cap$I$)$. So we have by $(*)$ $[\![\lambda y.xy]\!]_\rho^\triangledown{\in}[\![\bigcap_{i\in I}(A_i \to B_i)]\!]_F^\triangledown$. Since $\mathcal{I}_F^\triangledown$, $\mathcal{V}_F^\triangledown$ agree with $\nabla^\triangledown$, we get $[\![\bigcap_{i\in I}(A_i \to B_i)]\!]_F^\triangledown \subseteq [\![A]\!]_F^\triangledown$, so we conclude $[\![\lambda y.xy]\!]_\rho^\triangledown{\in}[\![A]\!]_F^\triangledown$. $\blacksquare$

17.2.27. THEOREM. *(Adequacy for the F-semantics) $\Gamma \models_F^\triangledown M : A$ implies $\Gamma \vdash^\triangledown M : A$ iff $\nabla^\triangledown$ is an F-type theory.*

PROOF. First we check that the given conditions are necessary.
Let $[\![x]\!]_\rho^\mathcal{D}{\in}[\![\psi \cap (A \to B)]\!]_\mathcal{V}^\mathcal{D}$ for some $\mathcal{I} = \langle \mathcal{D}, \cdot, [\![\;]\!]^\mathcal{D}, \Phi\rangle$, $\mathcal{V}$ which are F-$\triangledown$-good, agree with $\nabla^\triangledown$, and some good ρ. Then $[\![x]\!]_\rho^\mathcal{D}{\in}\Phi$, since $[\![A \to B]\!]_\mathcal{V}^\mathcal{D} \subseteq \Phi$. By Definition 17.2.8(ii) it follows $[\![\lambda y.xy]\!]_\rho^\mathcal{D}{\in}[\![\psi \cap (A \to B)]\!]_\mathcal{V}^\mathcal{D}$, hence $x{:}\psi \cap (A \to B) \models_F^\triangledown \lambda y.xy : \psi \cap (A \to B)$. By a similar argument we can obtain $x{:}\psi \cap \nu \models_F^\triangledown \lambda y.xy : \psi \cap \nu$ when $\nu{\in}\mathbb{C}^\triangledown$. Therefore we have F-adequacy of $\lambda\cap^\triangledown$ only if we can prove $x : \psi \cap (A \to B) \vdash^\triangledown \lambda y.xy : \psi \cap (A \to B)$ (respectively $x : \psi \cap \nu \vdash_\nu^\triangledown \lambda y.xy : \psi \cap \nu$ when $\nu{\in}\mathbb{C}^\triangledown$). Let $\nu \notin \mathbb{C}^\triangledown$.

$$x{:}\psi \cap (A \to B) \vdash^\triangledown \lambda y.xy : \psi \cap (A \to B) \quad \Rightarrow$$

$\Rightarrow$ $x{:}\psi \cap (A \to B), y{:}A_i \vdash^{\nabla} xy : B_i$
for some $I, A_i, B_i \in \mathbb{T}^{\nabla}$ such that $\bigcap_{i\in I}(A_i \to B_i) \leq_{\nabla} \psi \cap (A \to B)$,
by Theorem 15.1.2(**??**)
$\Rightarrow$ $x{:}\psi \cap (A \to B), y{:}A_i \vdash^{\nabla} x : C_i \to B_i$ and $x{:}\psi \cap (A \to B), y{:}A_i \vdash^{\nabla} y : C_i$
for some $C_i \in \mathbb{T}^{\nabla}$, by Theorem 15.1.16(**??**)
$\Rightarrow$ $\psi \cap (A \to B) \leq_{\nabla} \bigcap_{i\in I}(C_i \to B_i)$ and $A_i \leq_{\nabla} C_i$
by Theorem 15.1.16(**??**)
$\Rightarrow$ $\psi \cap (A \to B) \leq_{\nabla} \bigcap_{i\in I}(A_i \to B_i)$
by rule (η).

So we conclude $\psi \cap (A \to B) \sim_{\nabla} \bigcap_{i\in I}(A_i \to B_i)$.
Similarly from $x{:}\psi \cap \nu \models^{\nabla}_{F} \lambda y.xy : \psi \cap \nu$ we can show that either $\nu \leq_{\nabla} \psi$ or $\psi \cap \nu \sim_{\nabla} \bigcap_{i\in I}(A_i \to B_i)$ when $\nu \in \mathbb{C}^{\nabla}$.
For the vice versa, we consider the F-interpretation domain $\mathcal{I}^{\nabla}_{F}$ and the type environment $\mathcal{V}^{\nabla}_{F}$ of Definition 17.2.23. They are ∇-F-good and agree with ∇^{∇} by Lemma 17.2.26. By Lemma 17.2.25 $[\![A]\!]^{\nabla}_{F} = \{X \in \mathcal{F}^{\nabla} \mid A \in X\}$. So we have that $\Gamma \models^{\nabla}_{F} M : A$ implies $\Gamma \vdash^{\nabla} M : A$ mimicking the proof of Theorem 17.2.17. ∎

The type theories $\nabla^{\mathrm{HL}}, \nabla^{\mathrm{EHR}}, \nabla^{\mathrm{AO}}, \nabla^{\mathrm{Scott}}, \nabla^{\mathrm{Park}}, \nabla^{\mathrm{CDZ}}, \nabla^{\mathrm{HR}}$ are adequate for the F-semantics. Moreover for $\nabla^{\mathrm{Scott}}, \nabla^{\mathrm{Park}}, \nabla^{\mathrm{CDZ}}, \nabla^{\mathrm{HR}}$ the F-semantics coincides with the simple semantics. The remaining type theories of Figure 17.3 are not adequate for the F-semantics. The remark concerning λ-models and restricted λ-models made for the inference and the simple semantics, applies also to the F-semantics.

In the literature there are essentially five ways of interpreting Curry's types in a model of the untyped λ-calculus. They differ essentially in the interpretation of the *arrow* type constructor. We shall mainly follow the terminology of Hindley [1983a].

The simple and the F-semantics are defined as expected.

Following Scott [1980b], the *quotient set semantics* takes into account that we want to consider equivalent two functions iff they give equivalent results when applied to equivalent arguments. So types are interpreted as partial equivalence relations of the domain rather than simply as subsets. The arrow constructor is defined as for logical relations: $d \sim_{A\to B} d'$ iff for all c, c' such that $c \sim_A c'$ it holds $d \cdot c \sim_B d' \cdot c'$.

The *F-quotient set semantics* (Scott [1976]), modifies the quotient set semantics, in the same way as the F-semantics modifies the simple semantics. Namely it requires that all elements of the domain which are equivalent with respect to an arrow must be canonical representatives of functions.

Finally, Mitchell in Mitchell [1988] introduces another semantics, which he calls *inference semantics*, in which the interpretation of the arrow must *at least* contain the canonical representatives of functions which behave correctly with respect to the application.

All the above semantics easily extend to *intersection-types* (Barendregt et al. [1983]), (Dezani-Ciancaglini and Margaria [1986]) and to *polymorphic types* (Mitchell [1988]).

The crucial question in the semantics of types is the completeness of type assignment systems. Hindley proved in Hindley [1983a] that Curry's type assignment system is complete for all the mentioned semantics. More specifically Hindley [1983a] and Hindley [1983b] show the completeness for the simple semantics and moreover that:

1. $\Gamma \models_F^{\triangledown} M : A$ if and only if $\Gamma \models_s^{\triangledown} M : A$, when A is a Curry type;
2. the simple semantics is a particular case of the quotient set semantics;
3. the F-semantics is a particular case of the F-quotient set semantics.

The argument (due to M.Coppo) showing points (2) and (3) easily extends to intersection and polymorphic types, so for these type disciplines it is enough to discuss only completeness for the simple and the F-semantics to get completeness results for the quotiented versions. One could define also a quotient version of the inference semantics, but this would be treated similarly.

Historically, filter models had been introduced in Barendregt et al. [1983] precisely to achieve the completeness result for the type assignment system $\lambda\cap_\Omega^{\mathrm{BCD}}$. And this result, together with the conservativity of $\lambda\cap_\Omega^{\mathrm{BCD}}$, with respect to Curry's simple types, was used in Barendregt et al. [1983] to prove Scott's conjecture concerning the completeness of the set-theoretic semantics for simple types.

The completeness with respect to the simple semantics, of various intersection-type assignment systems, over λ-models, has been proved also in Hindley [1982], Coppo et al. [1984], Coppo et al. [1987], van Bakel [1992].

As far as the completeness with respect to the F-semantics of intersection-type assignment systems over λ-models, we can cite Dezani-Ciancaglini and Margaria [1986], Yokouchi [1994], Abramsky and Ong [1993]. In Dezani-Ciancaglini and Margaria [1986] the intersection-type theories which give λ-models where some filters are never interpretations of λ-abstractions and which are complete for the F-semantics are characterized. More specifically it is shown that an intersection-type theory $\nabla^{\triangledown}$ satisfies the previous conditions if and only if $\Omega \not\leq_{\triangledown} \Omega \to \Omega$, types are invariant under β-equality of subjects, and moreover the following rule (due to R.Hindley):

$$\text{(Hindley rule)}\quad \frac{\Gamma \vdash^{\triangledown} M : \psi \cap (\Omega^n \to \Omega)\ \ x_i \notin \mathrm{FV}(M)\ (1 \leq i \leq n)}{\Gamma \vdash^{\triangledown} \lambda x_1 \ldots x_n . M x_1 \ldots x_n : \psi}$$

for all $\psi \in \mathbb{C}^{\triangledown}$, is a derived rule (see Exercise 16.6.30).

Yokouchi in Yokouchi [1994] shows that if we add two suitable rules (quite similar to Hindley rule) and rules (β-*exp*), (β-*red*) to the intersection-type assignment system of Coppo et al. [1981], we obtain a type assignment system which is complete for the F-semantics.

Abramsky and Ong, in Abramsky and Ong [1993], prove the completeness of the theory ∇^{AO} with respect to the F-semantics over applicative structures with convergence.

We could have used the syntactic approach based on term models introduced in Hindley [1982] for showing all our adequacy results concerning the simple semantics, but not in a straightforward way as far as the inference or the F-semantics. To this end, notice that if $\rho(x) = [M]$ and $M\mathsf{I}$ reduces to an abstraction then a fortiori $M\mathbf{1}$ reduces to an abstraction. Therefore $[\![x\mathbf{1}]\!]_\rho$ is the representative of a function whenever $[\![x\mathsf{I}]\!]_\rho$ is the representative of a function. Now consider the F-type theory $(\{\varphi, \Omega\}, \mathcal{DM})$ where $\mathcal{DM} = \mathrm{AO} \cup \{\varphi \cap (\Omega \to \Omega) \sim \Omega \to \varphi\}$ (Dezani-Ciancaglini and Margaria [1986]). We have $\{x{:}(\varphi \to \varphi) \to \Omega \to \Omega\} \vdash_\Omega^{\mathcal{DM}} x\mathsf{I} : \Omega \to \Omega$ which implies, by soundness, $\{x{:}(\varphi \to \varphi) \to \Omega \to \Omega\} \models_F^{\mathcal{DM}} x\mathsf{I} : \Omega \to \Omega$. By the above we get $\{x{:}(\varphi \to \varphi) \to \Omega \to \Omega\} \models_F^{\mathcal{DM}} x\mathbf{1} : \Omega \to \Omega$, but it is easy to check, using the Generation Theorems I and II, that we cannot deduce $x\mathbf{1} : \Omega \to \Omega$ from $x{:}(\varphi \to \varphi) \to \Omega \to \Omega$. As a matter of fact, the proof of completeness for the F-semantics in Yokouchi [1994] uses a clever variant of the term model for a λ-calculus with constants. It is not clear to us if this could be adapted to the general case treated here.

Dezani-Ciancaglini et al. [2000] is an extended abstract of the present section.

17.3. Classical lambda models as filter models 26.2.2004:1018

In this section we give some applications of the results of section **??**, focusing on natural type structures and natural lambda-structures.

The first application is concerned with the lambda models D_∞, see Scott [1972] or Barendregt [1984]. These models are constructed from an initial D and it is not much of a restriction to consider them to be in **ALG**, i.e. the category of ω-algebraic lattices and Scott-continuous maps. To begin with, theorem 16.2.22 states that forall $\mathcal{D}\in$**NLS**

$$\mathcal{D} \cong \mathcal{F}^{\mathcal{K}(\mathcal{D})},$$

(isomorphism within the category **NLS**). For a given lambda model D_∞ obtained form a $D_0\in$**ALG** we have that $D_\infty\in$**NLS**. We can construct a $\mathcal{T}_\infty\in$**NTS** such that

$$K(D_\infty) \cong \mathcal{T}_\infty,$$

obtaining for such D_∞

$$D_\infty \cong \mathcal{F}^{\mathcal{T}_\infty}.$$

The construction of D_∞ does not only depend on the initial D_0, but also on the projection pair $\mathsf{i}_0, \mathsf{j}_0$ which gives the start of the D_∞ construction:

$$\mathsf{i}_0 : D_0 \to D_1,\ \mathsf{j}_0 : D_1 \to D_0,$$

where $D_1 = [D_0 \to D_0]$. By making various variations on the theme $(D_0, \mathsf{i}_0, \mathsf{j}_0)$ one obtains the following versions of D_∞

$$D_\infty^{\mathrm{Scott}}, D_\infty^{\mathrm{Park}}, D_\infty^{\mathrm{CDZ}}, D_\infty^{\mathrm{DHM}} \text{ and } D_\infty^{\mathrm{HR}},$$

each having some specific interesting property. For each of these versions D_∞^∇ there is a particular natural intersection typetheory $\mathcal{T}_\nabla$ such that

$$D_\infty^\nabla \cong \mathcal{F}^{T_\nabla}.$$

The pleasant fact is that $\mathcal{T}_\nabla$ and the triple $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$ correspond to each other in a canonical way. For $\nabla \in \{\mathrm{Scott}, \mathrm{Park}\}$ one has that the model D_∞^∇ was constructed first and the natural type theory $\mathcal{T}_\nabla$ came later. For $\nabla \in \{\mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$ one first constructed the natural type theory $\mathcal{T}_\nabla$ in order to obtain the model D_∞^∇ satisfying a certain property.

The second application illustrates the expressiveness and flexibility of intersection types: we give a new proof of the fact that the term $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is *easy* in the sense of Jacopini [1975], as has been shown in various ways, see e.g. Baeten and Boerboom [1979] or Mitschke's proof in Barendregt [1984], Proposition 15.3.9. Given any λ-term M, we inductively build natural intersection type structures $\mathcal{T}^n$ in such a way that the union of these structures, call it $\mathcal{T}^*$, forces the interpretation of M to coincide with the interpretation of Ω.

Further applications of intersection types consist of necessary conditions for filter λ-models to be *sensible* or *semi-sensible*. We will not consider this issue, see Zylberajch [1991]. Comment: Berline students?

D_∞-models

Since all ω-algebraic complete lattices which are extensional λ-models are clearly natural λ-structures, Theorem 16.2.22 implies that any such λ-model $\mathcal{D}$ is isomorphic to a filter λ-model $\mathcal{F}^{\mathcal{S}}$, provided that we have $\mathcal{S} \cong \mathcal{K}(\mathcal{D})$. In this section we show that in the special case of $\mathcal{D} = D_\infty$, one can obtain a concise type theoretic description of $\mathcal{K}(\mathcal{D}_\infty)$ via a suitable natural type structure $\mathcal{T}_\infty$. Remarkably this type theory $\mathcal{T}_\infty$ is exactly the natural type theory *freely generated* by defining $\mathbb{C}^\infty = \mathcal{K}(D_0)$ and, axiomatizing $\mathcal{T}_\infty$ by postulating on $\mathbb{T}^\infty = \mathbb{T}_\cap(\mathcal{K}(\mathcal{D}_0))$ the equalities which arise from encoding the initial embedding-projection pair $\langle \mathsf{i}_0, \mathsf{j}_0 \rangle$. In this section we follow Alessi [1991].

We fix some notations and recall the standard D_∞ construction.

17.3.1. Definition. (i) Let D_0 be an ω-algebraic complete lattice and

$$\langle \mathsf{i}_0, \mathsf{j}_0 \rangle$$

be an *embedding-projection* pair bewteen D_0 and $[D_0 \to D_0]$, i.e.

$$\begin{array}{lcl} \mathsf{i}_0 & : & D_0 \to [D_0 \to D_0] \\ \mathsf{j}_0 & : & [D_0 \to D_0] \to D_0 \end{array}$$

are Scott continuous maps satisfying

$$\begin{array}{lcl} \mathsf{i}_0 \circ \mathsf{j}_0 & \sqsubseteq & Id_{[D_0 \to D_0]} \\ \mathsf{j}_0 \circ \mathsf{i}_0 & = & Id_{D_0}. \end{array}$$

(ii) Define a *tower* $\langle \mathsf{i}_n, \mathsf{j}_n \rangle : D_n \to D_{n+1}$ in the following way:

- $D_{n+1} = [D_n \to D_n]$;
- $\mathsf{i}_n(f) = \mathsf{i}_{n-1} \circ f \circ \mathsf{j}_{n-1}$ for any $f \in D_n$;
- $\mathsf{j}_n(g) = \mathsf{j}_{n-1} \circ g \circ \mathsf{i}_{n-1}$ for any $g \in D_{n+1}$.

(iii) For $d:\mathbb{N}\to D$ write $d_n = d(n)$. The set D_∞ is defined by

$$D_\infty \quad = \quad \{d:\mathbb{N}\to D \mid \forall n{\in}\mathbb{N}.\ d_n{\in}D_n \ \&\ \ \mathsf{j}_n(d_{n+1}) = d_n\},$$

(iv) The ordering on D_∞ is given by

$$d \sqsubseteq e \quad \Leftrightarrow \quad \forall k{\in}\mathbb{N}.\ d_k \sqsubseteq e_k.$$

(v) Let $\langle \Phi_{m\infty}, \Phi_{\infty m}\rangle$ denote the standard embedding-projection pair from D_m to D_∞ defined as follows. For any $d^m{\in}D_m$, $d{\in}D_\infty$,

$\Phi_{m\infty}(d^m) = \langle\ \ldots, \mathsf{j}_{m-2}(\mathsf{j}_{m-1}(d^m)), \mathsf{j}_{m-1}(d^m), d^m, \mathsf{i}_m(d^m), \mathsf{i}_{m+1}(\mathsf{i}_m(d^m)), \ldots\ \rangle$,

$\Phi_{\infty m}(d) = d_m = d(m)$.

(vi) Let $\Phi_{mn} : D_m\to D_n$ be $\Phi_{\infty n}\circ\Phi_{m\infty}$.

17.3.2. Definition. Let

$$\begin{array}{rcl} F_\infty & : & D_\infty\to[D_\infty\to D_\infty] \\ G_\infty & : & [D_\infty\to D_\infty]\to D_\infty \end{array}$$

be defined as follows

$$\begin{array}{rcl} F_\infty(d)(e) & = & \bigsqcup_{n\in\mathbb{N}} \Phi_{n\infty}(d_{n+1}(e_n)); \\ G_\infty(f) & = & \bigsqcup_{n\in\mathbb{N}} \Phi_{(n+1)\infty}(\Phi_{\infty n}\circ f\circ\Phi_{n\infty}). \ \blacksquare \end{array}$$

17.3.3. Lemma. (i) $\mathsf{i}_n\circ\mathsf{j}_n \sqsubseteq Id_{[D_n\to D_n]}$, $\mathsf{j}_n\circ\mathsf{i}_n = Id_{D_n}$.

(ii) $\forall p,q{\in}D_n\ \mathsf{i}_{n+1}(p\Rightarrow q) = (\mathsf{i}_n(p)\Rightarrow\mathsf{i}_n(q))\ \&\ \mathsf{j}_{n+1}(\mathsf{i}_n(p)\Rightarrow\mathsf{i}_n(q)) = (p\Rightarrow q)$.

(iii) $\forall e{\in}\mathcal{K}(D_n)\ \mathsf{i}_n(e){\in}\mathcal{K}(D_{n+1})$.

(iv) $\forall e{\in}\mathcal{K}(D_n)\ [m\geq n\Rightarrow\Phi_{nm}(e){\in}\mathcal{K}(D_m)]$.

(v) *If $n\leq k\leq m$ and $d{\in}D_n$, $e{\in}D_k$, then*

$$\Phi_{nk}(d)\sqsubseteq e \ \Leftrightarrow\ \Phi_{nm}(d)\sqsubseteq\Phi_{km}(e)\ \Leftrightarrow\ \Phi_{n\infty}(d)\sqsubseteq\Phi_{k\infty}(e).$$

Proof. Easy. ■

17.3.4. Lemma. $\bigsqcup_{n\in\mathbb{N}}\Phi_{n\infty}\circ\Phi_{\infty n} = \mathbf{1}_{D_\infty}$

Proof. Since $\langle\Phi_{n\infty},\Phi_{\infty n}\rangle$ is an embedding-projection pair, we have forall $n{\in}\mathbb{N}$ $\Phi_{n\infty}\circ\Phi_{\infty n}\sqsubseteq \mathrm{Id}_{D_\infty}$, hence for all $d{\in}D_\infty$

$$\bigsqcup_{n\in\mathbb{N}}\Phi_{n\infty}\circ\Phi_{\infty n}(d)\sqsubseteq d.$$

On the other hand, for all $k{\in}\mathbb{N}$, we have

$$\begin{array}{rcl} (\bigsqcup_{n\in\mathbb{N}}\Phi_{n\infty}\circ\Phi_{\infty n}(d))_k & = & \bigsqcup_{n\in\mathbb{N}}\Phi_{n\infty}(d_k) \\ & \sqsupseteq & (\Phi_{k\infty}(d_k))_k \\ & = & d_k \end{array}$$

Therefore also

$$\bigsqcup_{n\in\mathbb{N}}\Phi_{n\infty}\circ\Phi_{\infty n}(d)\sqsupseteq d,$$

and we are done. ■

Next lemma characterizes the compact elements of D_∞ and $[D_\infty \to D_\infty]$.

17.3.5. LEMMA. (i) $d \in \mathcal{K}(D_\infty) \Leftrightarrow \exists k, e \in \mathcal{K}(D_k).\Phi_{k\infty}(e) = d.$

(ii) $f \in \mathcal{K}([D_\infty \to D_\infty]) \Leftrightarrow \exists k, g \in \mathcal{K}(D_{k+1}).f = \Phi_{k\infty} \circ g \circ \Phi_{\infty k}.$

(iii) *If* $f = \Phi_{k\infty} \circ g \circ \Phi_{\infty k}$ *with* $g \in D_{k+1}$, *then* $G_\infty(f) = \Phi_{k+1}(g)$.

PROOF. (i) ($\Rightarrow$) Let $d \in \mathcal{K}(D_\infty)$. Then $d = \bigsqcup_{n\in\mathbb{N}} \Phi_{n\infty}(d_n)$, by lemma 17.3.4. Since d is compact, there exists $k \in \mathbb{N}$ such that $d = \Phi_{k\infty}(d_n)$. Now we prove that $d_k \in \mathcal{K}(D_k)$. Let $X \subseteq D_k$ be directed. Then

$$
\begin{array}{rcll}
d_k \sqsubseteq \bigsqcup X & \Rightarrow & d \sqsubseteq \Phi_{k\infty}(\bigsqcup X) & \\
& \Rightarrow & d \sqsubseteq \bigsqcup \Phi_{k\infty}(X), & \text{since } \Phi_{k\infty} \text{ is continuous,} \\
& \Rightarrow & \exists x \in X.d \sqsubseteq \Phi_{k\infty}(x), & \text{for some } k \text{ since } d \text{ is compact,} \\
& \Rightarrow & \Phi_{\infty k}(d) \sqsubseteq \Phi_{\infty k} \circ \Phi_{k\infty}(x) & \\
& \Rightarrow & d_k \sqsubseteq x. &
\end{array}
$$

This proves that $d_k \in \mathcal{K}(D_k)$. ($\Leftarrow$) Easy.

(ii) ($\Rightarrow$) By lemma 17.3.4, we have

$$f = \bigsqcup_{n\in\mathbb{N}} \Phi_{n\infty} \circ (\Phi_{\infty n} \circ f \circ \Phi_{n\infty}) \circ \Phi_{\infty n}.$$

Using similar arguments as in the proof of (i), we have that

- $\exists k \in \mathbb{N}.f = \Phi_{k\infty} \circ (\Phi_{\infty k} \circ f \circ \Phi_{k\infty}) \circ \Phi_{\infty k}$,
- $(\Phi_{\infty k} \circ f \circ \Phi_{k\infty}) \in \mathcal{K}(D_{n+1})$.

Put $g = (\Phi_{\infty k} \circ f \circ \Phi_{k\infty})$. ($\Leftarrow$) Easy.

(iii) Let $f = \Phi_{k\infty} \circ g \circ \Phi_{\infty k}$ with $g \in D_{k+1}$. Then

$$
\begin{array}{rcll}
G_\infty(f) & = & G_\infty(\Phi_{k\infty} \circ g \circ \Phi_{\infty k}) & \\
& = & \bigsqcup_{n\in\mathbb{N}} \Phi_{(n+1)\infty}(\Phi_{\infty n} \circ (\Phi_{k\infty} \circ g \circ \Phi_{\infty k}) \circ \Phi_{n\infty}) & \\
& = & \bigsqcup_{n\in\mathbb{N}} \Phi_{(n+1)\infty}(\Phi_{kn} \circ g \circ \Phi_{nk}), & \text{by def. of } \Phi_{kn}, \\
& = & \bigsqcup_{n\in\mathbb{N}} \Phi_{(n+1)\infty}(\Phi_{(k+1)(n+1)}(g)), & \text{by def. of } \Phi_{kn}, \\
& = & \Phi_{(k+1)\infty}(g). \blacksquare &
\end{array}
$$

17.3.6. LEMMA. (i) $\bigsqcup X$ *exists for all* $X \subseteq D_\infty$.

(ii) $\forall x \in \mathcal{D}_\infty.x = \sqcup\{e \in \mathcal{K}(D_\infty) \mid e \sqsubseteq x\}$.

(iii) $\mathcal{K}(D_\infty)$ *is countable.*

(iv) $\mathcal{D}_\infty \in \mathbf{ALG}$.

PROOF. (i) Clearly $\bigsqcup X = \boldsymbol{\lambda} n \in \mathbb{N}.d^n$, where $d^n = \sqcup\{e_n \mid e \in X\}$.

(ii) Let $x \in D_\infty$ and let $U_x = \{e \in \mathcal{K}(D_\infty) \mid e \sqsubseteq x\}$. Clearly $U_x \sqsubseteq x$. Now let $U_x \sqsubseteq f \in D_\infty$ in order to show $x \sqsubseteq f$. Define

$$V = \{\Phi_{n\infty}(g^n) \mid g^n \in \mathsf{K}(D_n),\, g^n \sqsubseteq x_n\}.$$

Then by lemmas **??** and **??** one has $V \subseteq U$, hence $V \sqsubseteq f$. Let $V_n = \{g^n \in \mathcal{K}(D_n) \mid g^n \sqsubseteq x_n\}$. Then $V_n \sqsubseteq f_n$ and $x_n \sqsubseteq f_n$, since D_n is algebraic. Hence $x_n \sqsubseteq f$.

(iii) By Proposition ?? one has $D_n \in \mathbf{ALG}$ for each n. Hence $\mathcal{K}(D_n)$ is countable for each n. But then also $\mathcal{K}(D_\infty)$ is countable, by lemma 17.3.5(i).

(iv) By (i), (ii) and (iii). ∎

17.3.7. THEOREM. (Scott [1972]) *Let D_∞ be constructed from $D_0 \in \mathbf{ALG}$ and a projection pair $\mathsf{i}_0, \mathsf{j}_0$. Then $D_\infty \in \mathbf{ALG}$ and $\langle D_\infty, F_\infty, G_\infty \rangle$ is an extensional λ-model, i.e. $F_\infty \circ G_\infty = \mathbf{1}_{[D_\infty \to D_\infty]}$ and $G_\infty \circ F_\infty = \mathbf{1}_{D_\infty}$.* ∎

PROOF. See Barendregt [1984] theorem 18.2.16 for the proof that F and G are each other's inverse. ∎

17.3.8. COROLLARY. *Let D_∞ be constructed from $D_0 \in \mathbf{ALG}$ and a projection pair $\mathsf{i}_0, \mathsf{j}_0$. Then $\langle D_\infty, F_\infty, G_\infty \rangle$ is in* **NLS**.

PROOF. By lemma 17.3.6 (ii) we have $D_\infty \in \mathbf{ALG}$. Since it also is a λ-model, *a fortiori* $F_\infty \circ G_\infty \sqsupseteq \mathbf{1}_{[D_\infty \to D_\infty]}$ and $G_\infty \circ F_\infty \sqsubseteq \mathbf{1}_{D_\infty}$ and we are done. ∎

D_∞ as a filter λ-model

Let D_∞ be constructed from the triple $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$. To emphasize the dependency on t we write $D_\infty = D^t_\infty$. From the previous corollary and theorem 16.2.22 it follows that $D^t_\infty \cong \mathcal{F}^{\mathcal{K}(D^t_\infty)}$. In this subsection we transform $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$ into a (syntactic) intersection type structure $\mathcal{T}^t_\infty$ such that $\mathcal{T}^t_\infty \cong \mathcal{K}(D^t_\infty)$, hence

$$D^t_\infty \cong \mathcal{F}^{\mathcal{T}^t_\infty}.$$

17.3.9. DEFINITION. (Definition of $\mathcal{T}^t_\infty$) Let $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$ be given.

(i) The partial order $\leq_0$ on $\mathcal{K}(D_0)$ is defined by

$$d \leq_0 e \Leftrightarrow d \sqsupseteq e \quad d, e \in \mathcal{K}(D_0);$$

(ii) $\mathbb{T}^\infty = \mathcal{K}(D_0) \mid \mathbb{T}^\infty \to \mathbb{T}^\infty \mid \mathbb{T}^\infty \cap \mathbb{T}^\infty$.

(iii) Let $\leq_\infty$ be the natural type theory on $\mathbb{T}^\infty$ generated by the following extra axiom

$$c \cap d \sim_\infty c \sqcup d$$

and the extra rules

$$c \leq_0 d \;\Rightarrow\; c \leq_\infty d$$

and

$$i_o(e) = (c_1 \Rightarrow d_1) \sqcup \ldots \sqcup (c_n \Rightarrow d_n) \;\Rightarrow\; e \sim_\infty (c_1 \to d_1) \cap \ldots \cap (c_n \to d_n),$$

where $c, d, e, , c_1, d_1, \ldots, c_n, d_n \in \mathcal{K}(D_0)$.

(iv) $\mathcal{T}^t_\infty = \mathbb{T}^\infty / \sim_\infty$. ∎

The proof of the next lemma follows easily from Definition 17.3.9.

17.3.10. LEMMA. $c_1 \cap \ldots \cap c_n \sim_\infty c_1 \sqcup \ldots \sqcup c_n$. ∎

The proof of $\mathcal{T}^t_\infty \cong \mathcal{K}(D^t_\infty)$ will occupy 17.3.13-17.3.15. First we classify the types in $\mathbb{T}^\infty$ according to the maximal number of nested arrow occurrences they may contain.

17.3.11. Definition. (i) We define the map *rank* $rk : \mathbb{T}^\infty \to \mathbb{N}$ by:

$$\begin{array}{rcll} rk(c) & = & 0, & \text{for } c \in \mathcal{K}(D_0); \\ rk(A \to B) & = & \max\{rk(A), rk(B)\} + 1; & \\ rk(A \cap B) & = & \max\{rk(A), rk(B)\}. & \end{array}$$

(ii) Let $\mathbb{T}^\infty_n = \{A \in \mathbb{T}^\infty \mid rk(A) \le n\}$. ∎

We can associate to each type in $\mathbb{T}^\infty_n$ an element in D_n: this will be crucial for defining the required isomorphism (see Definition 17.3.16).

17.3.12. Definition. We define, for each $n \in \mathbb{N}$, a map $w_n : \mathbb{T}^\infty_n \to \mathcal{K}(D^t_n)$ by a double induction on n and on the construction of types in $\mathbb{T}^\infty$:

$$\begin{array}{lcl} w_n(c) & = & \Phi_{0n}(c); \\ w_n(A \cap B) & = & w_n(A) \sqcup w_n(B); \\ w_n(A \to B) & = & (w_{n-1}(A) \Rightarrow w_{n-1}(B)). \end{array}$$ ∎

17.3.13. Lemma. *Let $n \le m$ and $A \in \mathbb{T}^\infty_n$. Then $\Phi_{m\infty}(w_m(A)) = \Phi_{n\infty}(w_n(A))$.*

Proof. We show by induction on the definition of w_n that $w_{n+1}(A) = \mathsf{i}_n(w_n(A))$. Then the desired equality follows from the definition of the function Φ. The only interesting case is when $A \equiv B \to C$. We get

$$\begin{array}{rcll} w_{n+1}(B \to C) & = & w_n(B) \Rightarrow w_n(C), & \text{by definition,} \\ & = & \mathsf{i}_{n-1}(w_{n-1}(B)) \Rightarrow \mathsf{i}_{n-1}(w_{n-1}(C)), & \text{by induction,} \\ & = & \mathsf{i}_n(w_{n-1}(B) \Rightarrow w_{n-1}(C)), & \text{by Lemma 17.3.3(ii),} \\ & = & \mathsf{i}_n(w_n(B \to C)), & \text{by Definition 17.3.12.} \end{array}$$ ∎

The maps w_n reverse the order between types.

17.3.14. Lemma. *Let $n \ge rk(A \cap B)$. Then*

$$A \le_\infty B \;\Rightarrow\; w_n(B) \sqsubseteq w_n(A).$$

Proof. The proof is by induction on the definition of $\le_\infty$. We consider only two cases.

Case (η). Let $A \le_\infty B$ because $A \equiv C \to D$, $B \equiv E \to F$, $E \le_\infty C$ and $D \le_\infty F$. Then

$$\begin{array}{rcl} E \le_\infty C \;\&\; D \le_\infty F & \Rightarrow & w_{n-1}(C) \sqsubseteq w_{n-1}(E) \;\&\; w_{n-1}(F) \sqsubseteq w_{n-1}(D), \\ & & \text{by induction} \\ & \Rightarrow & w_{n-1}(E) \Rightarrow w_{n-1}(F) \sqsubseteq w_{n-1}(C) \Rightarrow w_{n-1}(D) \\ & \Rightarrow & w_n(B) \sqsubseteq w_n(A). \end{array}$$

Case $e \sim_\infty (c_1 \to d_1) \cap \ldots \cap (c_n \to d_n)$ as $i_o(e) = (c_1 \Rightarrow d_1) \sqcup \ldots \sqcup (c_n \Rightarrow d_n)$. We show by induction on $n \geq 1$ the following.

$$w_n(e) = (w_{n-1}(c_1) \Rightarrow w_{n-1}(d_1)) \sqcup \ldots \sqcup (w_{n-1}(c_k) \Rightarrow w_{n-1}(d_k)).$$

It trivially holds for $n = 1$, so let $n > 1$.

$$\begin{array}{lcl} w_n(e) & = & \mathsf{i}_{n-1}(w_{n-1}(e)) \\ & = & \mathsf{i}_{n-1}((w_{n-2}(c_1) \Rightarrow w_{n-2}(d_1)) \sqcup \ldots \sqcup (w_{n-2}(c_k) \Rightarrow w_{n-2}(d_k))) \\ & = & \mathsf{i}_{n-1}(w_{n-2}(c_1) \Rightarrow w_{n-2}(d_1)) \sqcup \ldots \sqcup \mathsf{i}_{n-1}(w_{n-2}(c_k) \Rightarrow w_{n-2}(d_k)) \\ & = & (\mathsf{i}_{n-2}(w_{n-2}(c_1)) \Rightarrow \mathsf{i}_{n-2}(w_{n-2}(d_1))) \sqcup \ldots \sqcup (\mathsf{i}_{n-2}(w_{n-2}(c_k)) \Rightarrow \mathsf{i}_{n-2}(w_{n-2}(d_k)) \\ & = & (w_{n-1}(c_1) \Rightarrow w_{n-1}(d_1)) \sqcup \ldots \sqcup (w_{n-1}(c_k) \Rightarrow w_{n-1}(d_k)). \blacksquare \end{array}$$

Also the reverse implication of Lemma 17.3.14 holds.

17.3.15. Lemma. *Let $rk(A \cap B) \leq n$. Then*

$$w_n(B) \sqsubseteq w_n(A) \;\Rightarrow\; A \leq_\infty B.$$

Proof. By induction on $rk(A \cap B)$.
If $rk(A \cap B) = 0$ we have $A \equiv \bigcap_{i \in I} c_i$, $B = \bigcap_{j \in J} d_j$. Then

$$\begin{array}{lcl} w_n(B) \sqsubseteq w_n(A) & \Rightarrow & \bigsqcup_{j \in J} \Phi_{0n}(d_j) \sqsubseteq \bigsqcup_{i \in I} \Phi_{0n}(c_i) \\ & \Rightarrow & \Phi_{n0}(\bigsqcup_{j \in J} \Phi_{0n}(d_j)) \sqsubseteq \Phi_{no}(\bigsqcup_{i \in I} \Phi_{0n}(c_i)) \\ & \Rightarrow & \bigsqcup_{j \in J} (\Phi_{n0} \circ \Phi_{0n})(d_j) \sqsubseteq \bigsqcup_{i \in I} (\Phi_{n0} \circ \Phi_{0n})(c_i) \\ & \Rightarrow & \bigsqcup_{j \in J} d_j \sqsubseteq \bigsqcup_{i \in I} c_i \\ & \Rightarrow & A \leq_\infty B. \end{array}$$

Otherwise, let

$$\begin{array}{lcl} A & \equiv & (\bigcap_{i \in I} c_i) \cap (\bigcap_{l \in L} (C_l \to D_l)), \\ B & \equiv & (\bigcap_{h \in H} d_h) \cap (\bigcap_{m \in M} (E_m \to F_m)). \end{array}$$

By Definition 17.3.9 we have

$$c_i \sim_\infty \bigcap_{j \in J_i} (a_j \to b_j), \;\; d_h \sim_\infty \bigcap_{k \in K_h} (e_k \to f_k).$$

Now for all $n \geq 1$

$$\begin{array}{lcl} w_n(c_i) & = & \bigsqcup_{j \in J_i} (w_{n-1}(a_j) \Rightarrow w_{n-1}(b_j))), \\ w_n(d_h) & = & (\bigsqcup_{k \in K_h} (w_{n-1}(e_k) \Rightarrow w_{n-1}(f_k))). \end{array}$$

So we get

$$\begin{array}{l}\bigsqcup_{h\in H}(\bigsqcup_{k\in K_h} w_{n-1}(e_k) \Rightarrow w_{n-1}(f_k)) \sqcup (\bigsqcup_{m\in M} w_{n-1}(E_m) \Rightarrow w_{n-1}(F_m)) \quad \sqsubseteq \\ \quad \bigsqcup_{i\in I}(\bigsqcup_{j\in J_i} w_{n-1}(a_j) \Rightarrow w_{n-1}(b_j)) \sqcup (\bigsqcup_{l\in L} w_{n-1}(C_l) \Rightarrow w_{n-1}(D_l)).\end{array}$$

Hence for each $h\in H$, $k\in K_h$ we have

$$\begin{array}{rcl} w_{n-1}(e_k) \Rightarrow w_{n-1}(f_k)) & \sqsubseteq & \bigsqcup_{i\in I}(\bigsqcup_{j\in J_i} w_{n-1}(a_j) \Rightarrow w_{n-1}(b_j)) \sqcup \\ & & (\bigsqcup_{l\in L} w_{n-1}(C_l) \Rightarrow w_{n-1}(D_l)). \end{array}$$

Suppose $w_{n-1}(f_k) \neq \perp_{D_n}$. Then there exist $I' \subseteq I$, $J_i' \subseteq J_i$, $L' \subseteq L$ such that

$$\begin{array}{rcl} \bigsqcup_{i\in I'}(\bigsqcup_{j\in J_i'} w_{n-1}(a_j)) \sqcup (\bigsqcup_{l\in L'} w_{n-1}(C_l)) & \sqsubseteq & w_{n-1}(e_k), \\ \bigsqcup_{i\in I'}(\bigsqcup_{j\in J_i'} w_{n-1}(b_j)) \sqcup (\bigsqcup_{l\in L'} w_{n-1}(D_l)) & \sqsupseteq & w_{n-1}(f_k). \end{array}$$

Since all types involved in the two above judgments have ranks strictly less than $rk(A\cap B)$, by induction and by Lemma 17.3.10 we obtain

$$\begin{array}{rcl} e_k & \leq_\infty & \bigcap_{i\in I'}(\bigcap_{j\in J_i'} a_j) \cap \bigcap_{l\in L'} C_l, \\ f_k & \geq_\infty & \bigcap_{i\in I'}(\bigcap_{j\in J_i'} b_j) \cap \bigcap_{l\in L'} D_l. \end{array}$$

Therefore we have $A \leq_\infty e_k \to f_k$.
If $w_{n-1}(f_k) = \perp_{D_n}$ then $A \leq_\infty e_k \to f_k$ follows with some effort by (Ω), $(\Omega\eta)$ and (η).
In a similar way we can prove that $A \leq_\infty E_m \to F_m$, for any $m\in M$. Putting together these results we get $A \leq_\infty B$. ∎

Remember that $\mathcal{K}(D_\infty) = (\mathcal{K}(D_\infty), \leq)$ where $\leq$ is the converse of $\sqsubseteq_{D_\infty}$. We can now prove the isomorphism in **NTS** between $\mathcal{K}(D_\infty)$ and $\mathcal{T}_\infty$

17.3.16. Definition. For $A\in\mathbb{T}^\infty$ write

$$\mathsf{m}([A]) = \Phi_{r\infty}(w_r(A)),$$

where $r \geq rk(A)$.

17.3.17. Theorem. *In* **NTS** *one has* $\mathcal{T}_\infty^t \cong \mathcal{K}(D_\infty^t)$ *via the map* m.

Proof. First of all notice that m is well defined, in that sense the it does not depend on either the type chosen in $[A]$ or the rank r. In fact let $B, B'\in[A]$, and let $p \geq rk(B)$, $p' \geq rk(B')$. Fix any $q \geq p, p'$. Then we have

$$\begin{array}{rcll} \Phi_{p\infty}(w_p(B)) & = & \Phi_{q\infty}(w_q(B)), & \text{by lemma 17.3.13,} \\ & = & \Phi_{q\infty}(w_q(B')), & \text{by lemma 17.3.14,} \\ & = & \Phi_{p'\infty}(w_{p'}(B')), & \text{by lemma 17.3.13.} \end{array}$$

Write $\mathsf{m}(A)$ for $\mathsf{m}([A])$. We prove that m satisfies all the conditions of Proposition 16.2.3. m is injective by Lemma 17.3.15 and monotone by lemma 17.3.14. It is surjective. In fact it is easily proved by induction on n that w_n is surjective on $\mathcal{K}(D_n)$,h ence m is surjective by Lemma 17.3.5(i). The function m^{-1} is monotone by lemma 17.3.15. Finally, we have to prove that m satisfies condition $(*)$ of proposition 16.2.3. Taking into account that the order $\leq$ on $\mathcal{K}(D^t_\infty)$ is the reversed of $\sqsubseteq$ of D^t_∞ and that, by definition **??**, we have $d \leadsto e = G_\infty(d \Rightarrow e)$, we can rephrase $(*)$ as:

$$(a) \qquad A \leq_\infty B \to C \quad \Leftrightarrow \quad \mathsf{m}(A) \sqsupseteq G_\infty(\mathsf{m}(B) \Rightarrow \mathsf{m}(C)).$$

In order to prove (a), let $r \geq max\{rk(A), rk(B \to C)\}$ (in particular it follows $rk(B), rk(C) \leq r-1$). We have

$$\begin{array}{rcll} G_\infty(\mathsf{m}(B) \Rightarrow \mathsf{m}(C)) & = & G_\infty(\Phi_{(r-1)\infty}(w_{r-1}(B)) \Rightarrow \Phi_{(r-1)\infty}(w_{r-1}(C))) & \\ & = & G_\infty(\Phi_{(r-1)\infty} \circ (w_{r-1}(B) \Rightarrow w_{r-1}(C)) \circ \Phi_{\infty(r-1)}), & \\ & & \text{by Lemma 17.3.3(ii)}, & \\ & = & \Phi_{r\infty}(w_{r-1}(B) \Rightarrow w_{r-1}(C)), \quad \text{by lemma 17.3.5(iii)}, & \\ & = & \Phi_{r\infty}(w_r(B \to C)), \quad \text{by definition of } w_r. & \end{array}$$

Finally we have

$$\begin{array}{rcll} A \leq_\infty B \to C & \Leftrightarrow & w_r(A) \sqsupseteq w_r(B \to C), & \\ & & \text{by lemmas 17.3.14 and 17.3.15}, & \\ & \Leftrightarrow & \Phi_{r\infty}(w_r(A)) \sqsupseteq \Phi_{r\infty}(w_r(B \to C)) & \\ & & \text{since } \Phi_{r\infty} \text{ is an embedding} & \\ & \Leftrightarrow & \Phi_{r\infty}(w_r(A)) \sqsupseteq G_\infty(\mathsf{m}(B) \Rightarrow \mathsf{m}(C)), & \text{as above}, \\ & \Leftrightarrow & \mathsf{m}(A) \sqsupseteq G_\infty(\mathsf{m}(B) \Rightarrow \mathsf{m}(C)). & \end{array}$$

So we have proved (a) and the proof is complete. ■

17.3.18. THEOREM. *$\mathcal{F}^{\mathcal{T}^t_\infty} \cong D^t_\infty$ in* **NLS**, *via the map*

$$\overline{\mathsf{m}}(X) = \bigcap\{m(B) \mid B{\in}X\}$$

satisfying $\underline{\mathsf{m}}(\uparrow A) = \mathsf{m}(A)$.

PROOF. One has $\mathcal{F}^{\mathcal{T}^t_\infty} \cong \mathcal{F}^{\mathcal{K}(D^t_\infty)} \cong D^t_\infty$, by theorems 17.3.17 and 16.2.21(i). Notice that if $\mathcal{T} \cong \mathcal{T}'$ via the map m, then $\mathcal{F}^{\mathcal{T}} \cong \mathcal{F}^{\mathcal{T}'}$ via the map

$$\overline{\mathsf{m}} = \{\mathsf{m}(X) \mid A{\in}X\}.$$

Hence

$$\begin{array}{rcl} \underline{\mathsf{m}}(X) & = & \bigsqcup \overline{\mathsf{m}}(X) \\ & = & \bigcap\{\mathsf{m}(B) \mid B{\in}X\} \\ & = & \mathsf{m}(A). \end{array}$$

In particular $\underline{\mathsf{m}}(\uparrow A) = \bigcap\{\mathsf{m}(B) \mid B \sqsupseteq A\} = \mathsf{m}(A)$.

Specific models D_∞ as filter models

In this subsection we specialize theorem 17.3.18 to D^t_∞ models constructed from specific triples $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$, five in total, each satisfying a specific modeltheoretic property. For each of these the corresponding type structure $\mathcal{T}^t_\infty$ can be described in a relatively simple way. This will be done as follows. Five type theories $\mathcal{T}_\nabla$ will be defined with $\nabla \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$. Also five corresponding choices of the triple $t_\nabla = (D_0, \mathsf{i}_0, \mathsf{j}_0)$ will be defined, giving rise to $D^{t_\nabla}_\infty$ again for $\nabla \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$. Moreover, we will show that $\mathcal{T}^{t_\nabla}_\infty = \mathcal{T}_\nabla$, hence

$$D^{t_\nabla}_\infty \cong \mathcal{F}^{\mathcal{T}^{t_\nabla}_\infty} \cong \mathcal{F}^{\mathcal{T}_\nabla},$$

by theorem 17.3.18. We will write $\mathcal{F}^\nabla := \mathcal{F}^{\mathcal{T}_\nabla}$, $D^\nabla_\infty := D^{t_\nabla}_\infty$ and $\mathcal{T}^\nabla_\infty := \mathcal{T}^{t_\nabla}_\infty$. Then the equation becomes

$$D^\nabla_\infty \cong \mathcal{F}^{\mathcal{T}^\nabla_\infty} \cong \mathcal{F}^\nabla.$$

17.3.19. Definition. (i) Remember the following type theoretic axioms.

(ω_{Scott})	$(\Omega{\to}\omega) \sim \omega$
(ω_{Park})	$(\omega{\to}\omega) \sim \omega$
$(\omega\varphi)$	$\omega \leq \varphi$
$(\varphi{\to}\omega)$	$(\varphi{\to}\omega) \sim \omega$
$(\omega{\to}\varphi)$	$(\omega{\to}\varphi) \sim \varphi$
(I)	$(\varphi{\to}\varphi) \cap (\omega{\to}\omega) \sim \varphi$

(ii) For $\nabla \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$ the following natural type theories $\mathcal{T}_\nabla$ have been defined by specifying its set of constants $\mathbb{C}$ and some extra axioms (besides the axioms $(\to\text{-}\cap)$, (Ω) and $(\Omega\eta)$ and the rule (η)).

∇	Constants $\mathbb{C}^\nabla$	Axioms of $\mathcal{T}_\nabla$
Scott	$\{\Omega, \omega\}$	$(\omega_{\mathrm{Scott}})$
Park	$\{\Omega, \omega\}$	$(\omega_{\mathrm{P}ark})$
CDZ	$\{\Omega, \omega, \varphi\}$	$(\omega\varphi), (\varphi{\to}\omega), (\omega{\to}\varphi)$
HR	$\{\Omega, \omega, \varphi\}$	$(\omega\varphi), (\varphi{\to}\omega), (I)$
DHM	$\{\Omega, \omega, \varphi\}$	$(\omega\varphi), (\omega{\to}\varphi), (\omega_{\mathrm{Scott}})$

(iii) As usual we write $\mathcal{T}_\nabla$ for $\mathbb{T}^\nabla / \sim_\nabla$.

17.3.20. Definition. (i) For $\nabla \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$ the following triples $(D_0, \mathsf{i}_0, \mathsf{j}_0)$ are defined. For $\nabla \in \{\mathrm{Scott}, \mathrm{Park}\}$, we define D^∇_0 as the two point chain $\{\Omega \sqsubseteq \omega\}$. For $\nabla \in \{\mathrm{CDZ}, \mathrm{HR}, \mathrm{DHM}\}$, we define D^∇_0 as the three point chain $\{\Omega \sqsubseteq \varphi \sqsubseteq \omega\}$.

(ii) We define $\mathsf{i}^\nabla_0 : D^\nabla_0 \to [D^\nabla_0 \to D^\nabla_0]$ and $\mathsf{j}^\nabla_0 : [D^\nabla_0 \to D^\nabla_0] \to D^\nabla_0$ as follows. First i^∇_0.

$$\mathsf{i}^\nabla_0(\Omega) \quad = \quad \Omega \Rightarrow \Omega \quad \text{for any } \nabla \in \{\mathrm{Scott}, \mathrm{Park}, \mathrm{CDZ}, \mathrm{DHM}, \mathrm{HR}\}$$

$$\mathsf{i}_0^{\bigtriangledown}(\varphi) \;=\; \begin{cases} (\varphi \Rightarrow \varphi) \sqcup (\omega \Rightarrow \omega) & \text{if } \bigtriangledown \in \{\mathrm{HR}\} \\ \omega \Rightarrow \varphi & \text{if } \bigtriangledown \in \{\mathrm{CDZ}, \mathrm{DHM}\} \end{cases}$$

$$\mathsf{i}_0^{\bigtriangledown}(\omega) \;=\; \begin{cases} \Omega \Rightarrow \omega & \text{if } \bigtriangledown \in \{\mathrm{Scott}, \mathrm{DHM}\} \\ \omega \Rightarrow \omega & \text{if } \bigtriangledown \in \{\mathrm{Park}\} \\ \varphi \Rightarrow \omega & \text{if } \bigtriangledown \in \{\mathrm{CDZ}, \mathrm{HR}\} \end{cases}$$

Then we define, for all $\bigtriangledown$, and $f \in [D_0^{\bigtriangledown} \to D_0^{\bigtriangledown}]$

$$\mathsf{j}_0^{\bigtriangledown}(f) \;=\; \bigsqcup\{d \in D_0^{\bigtriangledown} \mid \mathsf{i}_0^{\bigtriangledown}(d) \sqsubseteq f\}.$$

(iii) Write $t_{\bigtriangledown} = (D_0^{\bigtriangledown}, \mathsf{i}_0^{\bigtriangledown}, \mathsf{j}_0^{\bigtriangledown})$. It is easy to prove that $\langle \mathsf{i}_0^{\bigtriangledown}, \mathsf{j}_0^{\bigtriangledown} \rangle$ is an embedding-projection pair from $D_0^{\bigtriangledown}$ to $[D_0^{\bigtriangledown} \to D_0^{\bigtriangledown}]$, so we can build $D_\infty^{\bigtriangledown} = D_\infty^{t_{\bigtriangledown}}$ following the steps outlined in definition 17.3.1. ∎

17.3.21. LEMMA. *Let $\nabla \in \{Scott, Park, CDZ, DHM, HR\}$ and $c_1, \ldots, c_n, d_1, \ldots, d_n, e_1, \ldots, e_k, f_1, \ldots, f_k \in \mathcal{T}_{\bigtriangledown}$. Then*

$$(c_1 \Rightarrow d_1) \sqcup \ldots \sqcup (c_n \Rightarrow d_n) = (e_1 \Rightarrow f_1) \sqcup \ldots \sqcup (e_k \Rightarrow f_k) \;\Leftrightarrow$$
$$(e_1 \to f_1) \cap \ldots \cap (e_k \to f_k) \sim_{\bigtriangledown} (c_1 \to d_1) \cap \ldots \cap (c_n \to d_n).$$

PROOF. We need to prove

$$(c \Rightarrow d) \sqsubseteq (e_1 \Rightarrow f_1) \sqcup \ldots \sqcup (e_k \Rightarrow f_k) \;\Leftrightarrow\; (e_1 \to f_1) \cap \ldots \cap (e_k \to f_k) \leq_{\bigtriangledown} (c \to d).$$

Now, $(c \Rightarrow d) \sqsubseteq (e_1 \Rightarrow f_1) \sqcup \ldots \sqcup (e_k \Rightarrow f_k) \;\Leftrightarrow$

$$\begin{array}{lll} \Leftrightarrow & \exists I \subseteq \{1, \ldots, n\}\, [\sqcup_{i \in I} e_i \sqsubseteq c \;\&\; d \sqsubseteq \sqcup_{i \in I} f_i], & \text{by lemma 16.2.8} \\ \Leftrightarrow & \exists I \subseteq \{1, \ldots, n\}\, [c \leq_{\bigtriangledown} \cap_{i \in I} e_i \;\&\; \cup_{i \in I} f_i \leq_{\bigtriangledown} d], & \text{by beta-soundness, } (\eta) \\ & & \text{and } (\to\text{-}\cap), \\ \Leftrightarrow & (e_1 \Rightarrow f_1) \sqcup \ldots \sqcup (e_k \Rightarrow f_k) \leq_{\bigtriangledown} (c \to d). \;\blacksquare & \end{array}$$

17.3.22. COROLLARY. *The definition of $\mathsf{i}_0^{\bigtriangledown}$ is canonical. By this we mean that it could have been given by one definition, as the following holds.*

$$i_o^{\bigtriangledown}(e) = (c_1 \Rightarrow d_1) \sqcup \ldots \sqcup (c_n \Rightarrow d_n) \;\Leftrightarrow\; e \sim_{\bigtriangledown} (c_1 \to d_1) \cap \ldots \cap (c_n \to d_n).$$

PROOF. Immediate, by the definition of $\mathsf{i}_0^{\bigtriangledown}$, the axioms (Ω) and $(\Omega\eta)$, the special axioms $(\omega_{\mathrm{Scott}}), (\omega_{\mathrm{P}ark}), (\omega\varphi), (\varphi \to \omega), (\omega \to \varphi), (I)$ respectively, and the previous lemma. ∎

By Theorem 17.3.18 we get $\mathcal{F}^{\mathcal{T}_\infty^{t_{\bigtriangledown}}} \cong D_\infty^{t_{\bigtriangledown}}$ for these five cases, where $t_{\bigtriangledown}$ is the triple $t_{\bigtriangledown} = (D_0^{\bigtriangledown}, \mathsf{i}_0^{\bigtriangledown}, \mathsf{j}_0^{\bigtriangledown})$. We abbreviate this as $\mathcal{F}^{\mathcal{T}_\infty^{\bigtriangledown}} \cong D_\infty^{\bigtriangledown}$.

17.3.23. PROPOSITION. *For $\nabla \in \{Scott, Park, CDZ, DHM, HR\}$ one has*

$$\mathcal{T}_{\bigtriangledown} \cong \mathcal{T}_\infty^{\bigtriangledown}.$$

PROOF. Remember that

$$\begin{array}{rcl} \mathcal{T}_\nabla & = & \mathbb{T}(\mathbb{C}^\nabla)/\sim_\nabla \\ \mathcal{T}_\infty^\nabla & = & \mathbb{T}(\mathcal{K}(D_0^\nabla))/\sim_\infty . \end{array}$$

Then $\mathbb{T}(\mathbb{C}^\nabla) = \mathbb{T}(D_0^\nabla) = \mathbb{T}(\mathcal{K}(D_0^\nabla))$ for all ∇, since each D_0^∇ is finite. It remains to show that $\sim_\infty$ and $\sim_\nabla$ are equivalent. This follows from

$$A \leq_\infty B \;\Leftrightarrow\; A \leq_\nabla B.$$

As to $(\Rightarrow)$, this follows by induction on the generation of $\leq_\infty$. The theory $\mathcal{T}_\infty^\nabla$ satisfies the axioms $(\rightarrow\text{-}\cap), (\Omega)$ and is closed under the rule (η), since it is a natural type theory. It remains to show that the new axioms and rules are valid in this theory.

As to the axiom $c \leq_\infty d$ because $d \sqsubseteq c$, we have $d = \Omega, c = \omega$ or $d = c$. Then in all cases $c \leq_\nabla d$, by the axioms (Ω) and $(\omega\varphi)$.

As to the axiom $c \cap d \sim_\infty c \sqcup d$, with $c, d \in \mathcal{K}(D_0) = D_0$, we have, say, $c \sqsubseteq d$. Then $c \sqcup d = d$. Again we have $d \leq_\nabla c$. Therefore $d \leq_\nabla c \cap d \leq_\nabla d$, and hence $c \cap d \sim_\infty d = c \sqcup d$.

Finally, suppose $e \sim_\infty (c_1 \rightarrow d_1) \cap \ldots \cap (c_n \rightarrow d_n)$, because

$$\mathsf{i}_0(e) = (c_1 \Rightarrow d_1) \sqcup \ldots \sqcup (c_n \Rightarrow d_n).$$

Then $e \sim_\nabla (c_1 \rightarrow d_1) \cap \ldots \cap (c_n \rightarrow d_n)$, by lemma 17.3.21.

As to $(\Leftarrow)$, trivially $\mathcal{T}_\infty$ is closed under the rules of $\mathcal{T}_\nabla$. Moreover, all axioms of $\mathcal{T}_\nabla$ hold in $\mathcal{T}_\infty$, as follows from the definitions of i_0^∇. ■

Now we can obtain the following result

17.3.24. COROLLARY. *Let* $\nabla \in \{Scott, Park, CDZ, DHM, HR\}$. *Then in the category* **NLS** *we have*

$$\mathcal{F}^\nabla \cong D_\infty^\nabla.$$

PROOF.
$$\begin{array}{rcll} \mathcal{F}^\nabla & = & \mathcal{F}^{\mathcal{T}_\nabla}, & \text{by definition,} \\ & \cong & \mathcal{F}^{\mathcal{T}_\infty^\nabla}, & \text{by proposition 17.3.23,} \\ & \cong & D_\infty^\nabla, & \text{by Theorem 17.3.18. ■} \end{array}$$

We will end this subsection by telling what is the interest of the various models D_∞^∇. In Barendregt [1984], Theorem 19.2.9, the following result is proved.

17.3.25. THEOREM (Hyland and Wadsworth). *Let* $t = (D_0, \mathsf{i}_0, \mathsf{j}_0)$, *where* D_0 *is an arbitrary cpo (hence this applies to arbitrary elements of* **ALG**) *and*

$$\begin{array}{rcll} \mathsf{i}_0(d) & = & \boldsymbol{\lambda} e \in D_0.d, & \text{for } d \in \mathcal{D}_0, \\ \mathsf{j}_0(f) & = & f(\perp_{D_0}), & \text{for } f \in [D_0 \rightarrow D_0]. \end{array}$$

Then for $M, N \in \Lambda$ *(untyped lambda terms) and* $C[\,]$ *ranging over contexts*

$$D_\infty^t \models M = N \;\Leftrightarrow\; \forall C[\,].(C[M] \text{ is solvable} \;\Leftrightarrow\; C[N] \text{ is solvable}).$$

In particular, the local structure of D_∞^t *is independent of the initial* D_0. ■

17.3.26. COROLLARY. *For t as in the theorem one has for closed terms M, N*

$$D_\infty^t \models M = N \iff \forall A \in \mathcal{T}_{Scott} [\vdash^{Scott} M : A \iff \vdash^{Scott} N : A].$$

PROOF. Let $M, N \in \Lambda^\emptyset$. Then

$$\begin{aligned} D_\infty^t \models M = N \quad &\iff \quad D_\infty^{\text{Scott}} \models M = N, \text{ by Theorem 17.3.25,} \\ &\iff \quad \mathcal{F}^\nabla \models M = N, \text{ by Corollary 17.3.24,} \\ &\iff \quad \forall A \in \mathcal{T}_{\text{Scott}} [\vdash^{\text{Scott}} M : A \iff \vdash^{\text{Scott}} N : A], \end{aligned}$$

by Theorem **??**. ∎

The model D_∞^{Park} has been introduced to contrast the following result, see Barendregt [1984], 19.3.6.

17.3.27. THEOREM (Park). *Let t be as in 17.3.25. Then for the untyped λ-term*

$$\mathsf{Y}_{Curry} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

one has

$$[\![\mathsf{Y}_{Curry}]\!]^{D_\infty^t} = \mathsf{Y}_{Tarski},$$

where Y_{Tarski} is the least fixed-point combinator on D_∞^t. ∎

The model D_∞^{Park} has been constructed to be different.

17.3.28. THEOREM (Park). $[\![\mathsf{Y}_{Curry}]\!]^{D_\infty^{Park}} \neq \mathsf{Y}_{Tarski}$. ∎

Now this model can be obtained as a simple filter model $D_\infty^{\text{Park}} \cong \mathcal{F}^{\text{Park}}$ and therefore, by Corollory 17.3.24, one has

$$[\![\mathsf{Y}_{\text{Curry}}]\!]^{\mathcal{F}^{\text{Park}}} \neq \mathsf{Y}_{\text{Tarski}}.$$

The models D_∞^{Scott} and D_∞^{Park} have been introduced before the definition of the type theories $\mathcal{T}_{\text{Scott}}$ and $\mathcal{T}_{\text{Park}}$. In case of $D_\infty^{\text{CDZ}}, D_\infty^{\text{DHM}}$ and D_∞^{HR}, however, the natural type theories $\mathcal{T}_{\text{CDZ}}, \mathcal{T}_{\text{DHM}}$ and $\mathcal{T}_{\text{HR}}$ were defined first, in order to create models in which the images of important classes of lambda terms become definable.

17.3.29. PROPOSITION. *Let $M \in \Lambda^\emptyset$ be a closed lambda term.*

(i) *There exists a $d \in D_\infty^{CDZ}$ such that*

$$M \textit{ has a nf} \iff [\![M]\!]^{D_\infty^{CDZ}} \sqsupseteq d.$$

In fact we can take $d = \varphi$.

(ii) *There exists a $d \in D_\infty^{DHM}$ such that*

$$M \textit{ is solvable} \iff [\![M]\!]^{D_\infty^{DHM}} \sqsupseteq d.$$

In fact we can take $d = \varphi$.

(iii) *There exists a* $d \in D_\infty^{HR}$ *such that*

$$M \textit{ reduces to a } \lambda\mathsf{I}\textit{-term} \Leftrightarrow [\![M]\!]^{D_\infty^{HR}} \sqsupseteq d.$$

In fact we can take $d = \omega$.

PROOF. We explain the situation for (i), the other cases being similar.

$$\begin{array}{lll} M \text{has a nf} & \Leftrightarrow \; \vdash^{\mathrm{CDZ}} M : \varphi, & \text{by theorem 17.1.8,} \\ & \Leftrightarrow \; [\![M]\!]^{\mathcal{F}^{\mathrm{CDZ}}} \ni \varphi, & \text{by 16.4.4,} \\ & \Leftrightarrow \; [\![M]\!]^{\mathcal{F}^{\mathrm{CDZ}}} \sqsupseteq \uparrow\varphi, & \text{by the definition of filters,} \\ & \Leftrightarrow \; [\![M]\!]^{D_\infty^{\mathrm{CDZ}}} \sqsupseteq \underline{\mathsf{m}}(\uparrow\varphi) = \mathsf{m}(\varphi), & \text{by theorem 17.3.18,} \\ & \Leftrightarrow \; [\![M]\!]^{D_\infty^{\mathrm{CDZ}}} \sqsupseteq \Phi_{0,\infty}(\varphi), & \text{since } \varphi \in D_0, \\ & \Leftrightarrow \; [\![M]\!]^{D_\infty^{\mathrm{CDZ}}} \sqsupseteq \varphi, & \end{array}$$

by the identification D_0 as subset of D_∞. ■

As particular cases we get that Scott model as defined in Scott [1972] is isomorphic to the filter λ-model induced by the type theory ∇^{Scott} and Park model as defined in Park [1976] is isomorphic to the filter λ-model induced by the type theory ∇^{Park} (see Figure 17.3). This construction was first discussed in Coppo et al. [1984]. Other relevant references are Coppo et al. [1987], which presents the filter λ-model induced by the type theory ∇^{CDZ}, Honsell and Ronchi Della Rocca [1992], where the filter λ-models induced by the type theories $\nabla^{\mathrm{Park}}, \nabla^{\mathrm{HR}}$ and other models are considered, and Alessi [1991], Di Gianantonio and Honsell [1993], Plotkin [1993], where the relation between λ-structures and type theories is studied.

Results similar to Theorem 17.3.17 can be given also for other, non-extensional, inverse limit λ-models, obtained as solutions of domain equations involving also other functors. For instance one can consider $[\ \to\ ] \times A$, $[\ \to\ ] + A$, and also $[\ \to_\perp\ ] \times A$, and $[\ \to_\perp\ ] + A$, if one wants to analyze also models for restricted λ-calculi. In all such cases one gets concise type theoretic descriptions of the λ -models obtained as fixed points of such functors corresponding to suitable choices of G (Coppo et al. [1983]). Solutions of these equations will be discussed below. At least the following result is worthwhile mentioning in this respect, see Coppo et al. [1984] for a proof.

17.3.30. PROPOSITION. *The filter λ-model induced by ∇^{BCD} is isomorphic to $\langle \mathcal{D}, F, G\rangle$, where $\mathcal{D}$ is the initial solution of the domain equation $[\mathcal{D} \to \mathcal{D}] \times \mathsf{P}(\mathbb{C}_\infty) \equiv \mathcal{D}$, the pair $\langle F, G\rangle$ set u p a Galois connection*[1] *and G is the map which picks always the minimal element in the extensionality classes of all functions.* ■

[1] $F \circ G \sqsupseteq Id_{[\mathcal{D}\to\mathcal{D}]}$ & $G \circ F \sqsubseteq Id_{\mathcal{D}}$.

Graph λ-models as filter λ-models

In this section we give filter λ-models isomorphic to the classical *graph* model i.e. Scott's $\mathsf{P}(\omega)$ model (Scott [1972]), to Plotkin's model (Plotkin [1993]), and to Engeler's model D_A (Engeler [1981], Longo [1983]). The proofs of these isomorphisms are similar and simpler than that of Theorem 17.3.17, and we will leave them out, giving appropriate references and exercises.

The interest of such type theoretic accounts arises also from the fact that such models are not natural λ-structures. It is straightforward to see, in fact, that in these models the representatives of compact functions are not compact.

Remember the following rule for type theories.

$$\boxed{(\eta^{\sim})\quad \frac{A\sim A' \quad B\sim B'}{(A{\to}B)\sim(A'\sim B')}}$$

The following type theories are instrumental for defining the graph models.

$\triangledown$	Constants $\mathbb{C}^{\triangledown}$	Rules	Axioms of $\mathcal{T}_{\triangledown}$
Engeler	$\{\Omega,\varphi_0,\varphi_1,\ldots\}$	$-$	$(\Omega),(\Omega\eta)$
Plotkin	$\{\Omega,\omega\}$	$(\eta^{\sim})$	$(\Omega),(\to\text{-}\cap)$

Scott's $\mathsf{P}(\omega)$ model

Following the original notation by Scott, in the $\mathsf{P}(\omega)$ model ω denotes the set of natural numbers. $\mathsf{P}_{\mathrm{fin}}(\omega)$ denotes the set of finite subsets of ω.

Notation. (i) Let $\boldsymbol{\lambda} nm.\langle n,m\rangle:\mathbb{N}\times\mathbb{N}{\to}\mathbb{N}$ be a bijection, e.g. the well-known one defined by $\langle n,m\rangle=\frac{1}{2}(n+m)(n+m+1)+m$.

(ii) Let $\boldsymbol{\lambda} n.e_n:\mathbb{N}{\to}\mathsf{P}_{\mathrm{fin}}(\omega)$ be a bijection, e.g. the well-known one defined by

$$e_n=\{k_0,\ldots,k_{m-1}\} \text{ with } k_0<k_1<\ldots<k_{m-1} \;\Leftrightarrow\; n=\Sigma_{i<m}2^i.$$

17.3.31. Definition (Scott [1972]). Let $\gamma:\mathsf{P}_{\mathit{fin}}(\omega)\times\omega{\to}\omega$ be the bijection defined by

$$\gamma(e_n,m)=\langle n,m\rangle.$$

(i) Define $F_\omega:\mathsf{P}(\omega){\to}[\mathsf{P}(\omega){\to}\mathsf{P}(\omega)]$ by

$$F_\omega(X)=\bigsqcup\{u\Rightarrow i\mid\gamma(u,i){\in}X\}.$$

(ii) $G_\omega:[\mathsf{P}(\omega){\to}\mathsf{P}(\omega)]{\to}\mathsf{P}(\omega)$ by

$$G_\omega(f)=\{\gamma(u,i)\mid i{\in}f(u)\}$$

for all $f{\in}[\mathsf{P}(\omega){\to}\mathsf{P}(\omega)]$. ∎

17.3.32. Proposition. *Define for $X,Y{\in}\mathsf{P}(\omega)$ the application*

$$X\cdot_{\mathsf{P}(\omega)}Y=\{m\mid\exists e_n\subseteq Y\ \langle n,m\rangle{\in}X\}.$$

Then $F_\omega(X)(Y)=X\cdot_{\mathsf{P}(\omega)}Y$ is a (more common) equivalent definition for F_ω.

PROOF. Do exercise**??**. ∎

17.3.33. THEOREM (Scott [1972]). *$\mathsf{P}(\omega)$ is a λ-model via F_ω, G_ω.*

17.3.34. THEOREM (Alessi [1991]). *Define*

$$\begin{array}{lcl} \mathbb{C}^{\nabla^\omega} & = & \omega \\ \nabla^\omega & = & \nabla^{Engeler} \cup \{\bigcap_{k\in e}(k\to n) \sim \gamma(e,n) \mid e\in\mathsf{P}_{fin}, n\in\omega\}. \end{array}$$

Then $\mathsf{P}(\omega) \cong \mathcal{F}^{\nabla^\omega}$. ∎

Plotkin's Model

17.3.35. DEFINITION (Plotkin [1993]). Let φ be an atom.

(i) Define Pm as the least set such that

$$\mathsf{Pm} = \{\varphi\} \cup (\mathsf{P}_{fin}(\mathsf{Pm}) \times \mathsf{P}_{fin}(\mathsf{Pm})).$$

(ii) Define $\begin{array}{lcl} F_{\mathsf{Pm}} & : & \mathsf{P}(\mathsf{Pm})\to[\mathsf{P}(\mathsf{Pm})\to\mathsf{P}(\mathsf{Pm})] \\ G_{\mathsf{Pm}} & : & [\mathsf{P}(\mathsf{Pm})\to\mathsf{P}(\mathsf{Pm})]\to\mathsf{P}(\mathsf{Pm}) \end{array}$ by

$$\begin{array}{lcl} F_{\mathsf{Pm}}(X) & = & \bigsqcup\{u \Rightarrow v \mid \langle u,v\rangle\in X\} \\ G_{\mathsf{Pm}}(f) & = & \{\langle u,v\rangle \mid v \subseteq f(u)\}. \end{array}$$ ∎

17.3.36. THEOREM (Plotkin [1993]). *$F_{\mathsf{Pm}}, G_{\mathsf{Pm}}$ satisfy $F_{\mathsf{Pm}}\circ G_{\mathsf{Pm}} = Id$, making $\mathsf{P}(\mathsf{Pm})$ a λ-model.* ∎

17.3.37. THEOREM (Plotkin [1993]). *Let $\nabla_{Plotkin}$ be as defined in Figure 17.3. Then $\mathsf{P}(\mathsf{Pm}) \cong \mathcal{F}^{\nabla_{Plotkin}}$ are isomorphic as natural λ-structures (λ-models).* ∎

Engeler's Model

17.3.38. DEFINITION (Engeler [1981]). Let $\mathbb{C}_\infty$ be a countable set of atoms.

(i) Define Em as the least set satisfying $\mathsf{Em} = \mathbb{C}_\infty \cup (\mathsf{P}_{fin}(\mathsf{Em}) \times \mathsf{Em})$

(ii) Define $\begin{array}{lcl} F_{\mathsf{Em}} & : & \mathsf{P}(\mathsf{Em}) \to [\mathsf{P}(\mathsf{Em})\to\mathsf{P}(\mathsf{Em})] \\ G_{\mathsf{Em}} & : & [\mathsf{P}(\mathsf{Em})\to\mathsf{P}(\mathsf{Em})]\to\mathsf{P}(\mathsf{Em}) \end{array}$ by

$$\begin{array}{lcl} F_{\mathsf{Em}}(X) & = & \bigsqcup\{u \Rightarrow e \mid \langle u,e\rangle\in X\} \\ G_{\mathsf{Em}}(f) & = & \{\langle u,e\rangle \mid e\in f(u)\}. \end{array}$$ ∎

17.3.39. THEOREM (Engeler [1981]). *$F_{\mathsf{Em}}, G_{\mathsf{Em}}$ satisfy $F_{\mathsf{Em}}\circ G_{\mathsf{Em}} = \mathbf{1}$, making $\mathsf{P}(\mathsf{Em})$ a λ-model.* ∎

17.3.40. THEOREM (Plotkin [1993]). *Let $\nabla_{Engeler}$ be as defined in Figure 17.3. Then $\mathsf{P}(\mathsf{Em}) \cong \mathcal{F}^{\nabla_{Engeler}}$ are isomorphic as natural λ-structures (λ-models).* ∎

Lazy filter models

Intersection types are flexible enough to allow for the description of λ-models which are computationally adequate for the lazy operational semantics (Abramsky and Ong [1993]), i.e. *lazy* λ-models. For instance we have the easy result:

17.3.41. THEOREM. *Let ∇ be a natural λ-structures, not satisfying $(\Omega\eta)$, that is such that all unsolvables of order zero*[2] *have only type Ω. Then $\mathcal{F}^{\nabla}$ is a lazy λ-models.* ■

One of the simplest type theories one can think of is ∇^{AO} as defined in Figure 17.3. This gives a lazy λ-model, which is discussed in Abramsky and Ong [1993]. It can be also used for proving the completeness of *F-semantics* (see Dezani-Ciancaglini and Margaria [1986]).

17.3.42. THEOREM (Abramsky and Ong [1993]). *Let D_∞^{lazy} be the initial solution of the domain equation $D \cong [D \to D]_\perp$ in* **ALG**. *Then $D_\infty^{lazy} \cong \mathcal{F}^{AO}$.* ■

Filter λ-model for restricted lambda calculi

The call-by-value λ-calculus.

Models of call-by-value λ-calculus are considered in Egidi et al. [1992], Pravato [n.d.], Pravato et al. [n.d.].

Corollary 15.2.8(i) implies that all filter structures induced by type theories which contain ν and validate β_v-reduction are models of the call-by-value λ-calculus with the proviso that type environments must associate non empty filters to term variables.

The filter structure induced by the type theory ∇^{EHR} of Figure 17.3 gives a *fully abstract* model for the *call-by-value* operational semantics (for a proof see Egidi et al. [1992]). Moreover one has the following.

17.3.43. THEOREM (Egidi et al. [1992]). *Let D_∞^{strict} be an inverse limit solution of the domain equation $D \cong [D \to_\perp D]_\perp$. Then $D_\infty^{strict} \cong \mathcal{F}^{\nabla_{EHR}}$.* ■

*The λ-**I**-N-calculus.*

Models of the λ-**I** and the λ-N-calculi are considered in Honsell and Lenisa [1993], Honsell and Lenisa [1999]. These are λ-applicative structures with interpretations which equate all β-**I** or β-N-redexes to their contracta. So all filter structures induced by type theories which validate β-**I** and β-N-reduction are models of the λ-**I**-N-calculus. Like Theorem 17.3.17 one has the following.

17.3.44. THEOREM (Honsell and Lenisa [1999]). *Let $D_\infty^{!}$ be the inverse limit solutions of the domain equation $[D \to_\perp D] \cong D$. Then $D_\infty^{!} \cong \mathcal{F}^{\nabla}$, where ∇ is a Ω-free natural type theory, with $\mathbb{C}^{\nabla} = \mathcal{K}(D_0)$.*

Honsell and Lenisa [1999] discusses a filter structure which gives a *computational ly adequate* model for the *perpetual* operational semantics and a mathematical model for the maximal sensible λ-**I**-theory.

[2] A term is of order zero iff it cannot reduce to an abstraction.

A filter model equating an arbitrary closed term to $\Delta\Delta$

In Jacopini [1975] it has been proved by an analysis of conversion that the lambda term $\Delta\Delta$, where $\Delta = \lambda x.xx$, is easy, i.e. for any closed lambda term M the equation $\Delta\Delta = M$ is consistent. This fact was proved by a Church-Rosser argument by Mitschke, see Mitschke [1976] or Barendregt [1984], proposition 15.3.9. A model theoretical proof was given by Baeten and Boerboom [1979] who showed that for any closed M one has

$$\mathcal{P}(\omega) \models \Delta\Delta = M,$$

for a particular way of coding pairs on the set of natural numbers ω. We will now present the proof in Alessi et al. [2001] using intersection types. For an arbitrary closed λ-term M we will build a filter model $\mathcal{F}^{\nabla_M}$ such that

$$\mathcal{F}^{\nabla_M} \models \Delta\Delta = M.$$

We first examine which types can be assigned to Δ and $\Delta\Delta$.

17.3.45. Lemma. *Let $\mathcal{T}^{\triangledown}$ be a natural type structure that is β-sound.*
(i) $\vdash^{\triangledown} \Delta : A \to B \;\Leftrightarrow\; A \leq_{\triangledown} A \to B$.
(ii) $\vdash^{\triangledown} \Delta\Delta : B \;\Leftrightarrow\; \exists A{\in}\mathbb{T}^{\triangledown}. \vdash^{\triangledown} \Delta : A \leq_{\triangledown} (A \to B)$.
(iii) $\vdash^{\triangledown} \Delta\Delta : B \;\Leftrightarrow\; \exists A{\in}\mathbb{T}^{\triangledown}. A \leq_{\triangledown} (A \to B)$.

Proof. (i) ($\Leftarrow$) Suppose $A \leq (A{\to}B)$. Then

$$\begin{array}{rcl} x{:}A & \vdash & x : (A{\to}B) \\ x{:}A & \vdash & xx : B \\ & \vdash & \lambda x.xx : (A{\to}B). \end{array}$$

($\Rightarrow$) Suppose $\vdash \Delta : (A \to B)$. If $B = \Omega$, then $A \leq \Omega = (A \to B)$, by proposition 15.1.8(ii). Otherwise, by theorem 15.1.16,

$$\begin{array}{rlll} \vdash \lambda x.xx : (A{\to}B) & \Rightarrow & x{:}A \vdash xx : B, & \\ & \Rightarrow & x{:}A \vdash x : C, \quad x{:}A \vdash x : (C{\to}B), & \text{for some } C, \\ & \Rightarrow & A \leq (C{\to}B) \leq (A{\to}B), & \text{by } (\eta). \end{array}$$

(ii) ($\Leftarrow$) Immediate. ($\Rightarrow$) If $B = \Omega$, then $\vdash \Delta : \Omega \leq \Omega{\to}B$. If $B \neq \Omega$, then by Theorem 15.1.16(ii) one has $\vdash \Delta : (A{\to}B)$, $\vdash \Delta : A$, for some A. By (i) one has $A \leq A{\to}B$.
(iii) By (i) and (ii). ■

We associate to each type the maximum number of nested arrows in the leftmost path.

17.3.46. Definition. Let $\triangledown$ be a type theory. For $A{\in}\mathbb{T}^{\triangledown}$ its *type nesting*, notation $\#(A)$, is defined inductively on types as follows:

$$\begin{array}{lcll} \#(A) & = & 0 & \text{if } A{\in}\mathbb{C}^{\triangledown}; \\ \#(A \to B) & = & \#(A) + 1; & \\ \#(A \cap B) & = & \max\{\#(A), \#(B)\}. & \end{array}$$

Lemma 17.3.45(ii) can be strengthened using type nesting. First we need the following lemma that shows that in a strong beta type theory, Definition 15.1.10, any type A with $\#(A) \geq 1$ is equivalent to an intersection of arrows with the same type nesting.

17.3.47. LEMMA. *Let ∇ be strong β. Then for all $A\in\mathbb{T}^{\nabla}$ with $\#(A) \geq 1$, there exists an $A' \equiv (\bigcap_{i\in I}(C_i \to D_i)) \sim_{\nabla} A$ such that $\#(A') = \#(A)$.*

PROOF. Every type A is an intersection of arrow types and constants:

$$A \equiv (C \to B) \cap \ldots \cap \psi \cap \ldots.$$

Since ∇ is strongly β, the constants can be replaced by an intersection of arrows between constants. As $\#(A) \geq 1$ this does not increase the type nesting. ■

17.3.48. LEMMA. *Let ∇ be a natural type theory which is strongly beta. Then*

$$\vdash^{\nabla} \Delta\Delta : B \;\Rightarrow\; \exists A\in\mathbb{T}^{\nabla}[\vdash^{\nabla} \Delta : A \leq_{\nabla} A \to B \;\&\; \#(A) = 0].$$

PROOF. Let $\vdash \Delta\Delta : B$. If $B \sim \Omega$ take $A \equiv \Omega$. Otherwise, by Lemma 17.3.45(ii), there exists $A\in\mathbb{T}^{\nabla}$ such that $\vdash \Delta : A$ and $A \leq A \to B$. We show by course of value induction on $n = \#(A)$ that we can take an alternative A' with $\#(A') = 0$. If $n = 0$ we are done, so suppose $n \geq 1$. By Lemma 17.3.47, we may assume that A is of the form $A \equiv (C_1 \to D_1) \cap \ldots \cap (C_m \to D_m)$. Now $A \leq A \to B$, hence $A \leq C_{i_1} \cap \ldots \cap C_{i_p}$ and $D_{i_1} \cap \ldots \cap D_{i_p} \leq B$, with $1 \leq i_1, \ldots, i_p \leq m$, since ∇ is beta. Since $B \neq \Omega$ one has $p > 0$. Hence,

$$\begin{array}{lll}
\vdash \Delta : A & \Rightarrow \;\; \vdash \Delta : (C_{i_k} \to D_{i_k}), & 1 \leq k \leq p, \\
& \Rightarrow \;\; C_{i_k} \leq (C_{i_k} \to D_{i_k}), & \text{by 17.3.45(i)}, \\
& \Rightarrow \;\; \bigcap_k C_{i_k} \leq \bigcap_k (C_{i_k} \to D_{i_k}) & \\
& \qquad \leq (\bigcap_k C_{i_k} \to \bigcap_k D_{i_k}), & \text{since } \nabla \text{ is natural}, \\
& \qquad \leq (\bigcap_k C_{i_k} \to B), & \text{as } \bigcap_k D_{i_k} \leq B.
\end{array}$$

Now take $A' = \bigcap_k C_{i_k}$. Then $\#(A') < n$ and we are done by the IH. ■

Now let $M\in\Lambda^{\emptyset}$. We will build the desired model satisfying $\models \Delta\Delta = M$ by taking the union of a countable sequence of type theories ∇_n defined in a suitable way to force the final interpretation of M to coincide with the interpretation of $\Delta\Delta$. In the following $\langle\cdot,\cdot\rangle$ denotes any bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

17.3.49. DEFINITION. (i) Define the growing sequence of intersection type theories ∇_n by induction on $n\in\mathbb{N}$, specifying the constants, axioms and rules.

- $\mathbb{C}^{\nabla_0} = \{\Omega, \omega\}$;
- $\nabla_0 = \{(\to\text{-}\cap), (\Omega), (\Omega\eta), (\eta)\} \cup \{\omega_{\text{Scott}}\}$;
- $\mathbb{C}^{\nabla_{n+1}} = \mathbb{C}^{\nabla_n} \cup \{\xi_{\langle n,m\rangle} \mid m\in\mathbb{N}\}$;
- $\nabla_{n+1} = \nabla_n \cup \{\xi_{\langle n,m\rangle} \sim (\xi_{\langle n,m\rangle} \to W_{\langle n,m\rangle})\}$,

where $\langle W_{\langle n,m\rangle}\rangle_{m\in\mathbb{N}}$ is any enumeration of the set

$$\{A \mid \vdash^{\nabla_n} M : A\}.$$

(ii) We define ∇_M as follows:

$$\mathbb{C}^{\nabla_M} = \bigcup_{n\in\mathbb{N}} \mathbb{C}^{\nabla_n}; \quad \nabla_M = \bigcup_{n\in\mathbb{N}} \nabla_n.$$

17.3.50. Proposition. *$\mathcal{T}^{\nabla_M}$ is a β-sound natural intersection type theory.*

Proof. Clearly the axioms $(\Omega), (\Omega\eta)$ and $(\to\text{-}\cap)$ are valid in $\mathcal{T}^{\nabla_M}$, as they are already in $\mathcal{T}^{\nabla_0}$. The validity of rule (η) in $\mathcal{T}^{\nabla_M}$ follows by a "compactness" argument: if $A' \leq_{\nabla_M} A$ & $B \leq_{\nabla_M} B'$, then $A' \leq_{\nabla_n} A$ & $B \leq_{\nabla_m} B'$; but then $(A\to B) \leq_{\nabla_{\max(n,m)}} (A'\to B')$ and hence $(A\to B) \leq_{\nabla_M} (A'\to B')$. Therefore the type theory is natural. By a similar compactness argument $\mathcal{T}^{\nabla_M}$ is β-sound. ∎

17.3.51. Theorem. *The filter structure $\mathcal{F}^{\nabla_M}$ is an extensional λ-model.*

Proof. By Corollary **??** $\mathcal{F}^{\nabla_M}$ is a λ-model. Moreover, ∇_M is a natural eta type theory, so by Theorem **??** extensionality follows. ∎

We now need to show that some types cannot be deduced for Δ.

17.3.52. Lemma. *$\not\vdash^{\nabla_M} \Delta : \omega$ and $\not\vdash^{\nabla_M} \Delta : (\omega \to \omega) \to \omega \to \omega$.*

Proof. Define the set $\mathcal{E}_\Omega \subseteq \mathbb{T}(\mathbb{C}^{\nabla_M})$ as the minimal set such that:

$$\begin{array}{lll}
\Omega\in\mathcal{E}_\Omega; & & \\
A\in\mathbb{T}^{\nabla_M}, B\in\mathcal{E}_\Omega & \Rightarrow & (A \to B)\in\mathcal{E}_\Omega; \\
A, B\in\mathcal{E}_\Omega & \Rightarrow & (A \cap B)\in\mathcal{E}_\Omega; \\
W_i\in\mathcal{E}_\Omega & \Rightarrow & \xi_i\in\mathcal{E}_\Omega.
\end{array}$$

Claim: $A\in\mathcal{E}_\Omega \Leftrightarrow A \sim_{\nabla_M} \Omega$.

($\Rightarrow$) By induction on the definition of $\mathcal{E}_\Omega$, using Lemma 15.1.8(i).

($\Leftarrow$) By induction on $\leq_{\nabla_M}$ it follows that

$$\mathcal{E}_\Omega \ni B \leq_{\nabla_M} A \;\Rightarrow\; A\in\mathcal{E}_\Omega.$$

Hence if $A \sim_{\nabla_M} \Omega$, one has $\mathcal{E}_\Omega \ni \Omega \leq_{\nabla_M} A$ and thus $A\in\mathcal{E}_\Omega$.

As $\omega\notin\mathcal{E}_\Omega$, it follows by the claim that

$$\Omega \not\sim_{\nabla_M} \omega. \tag{+}$$

Suppose towards a contradiction that $\vdash^{\nabla_M} \Delta : \omega$. Then $\vdash^{\nabla_M} \Delta : \Omega \to \omega$, by ω_{Scott}. By Lemma 17.3.45(i) we get $\Omega \leq_{\nabla_M} (\Omega \to \omega) \sim_{\nabla_M} \omega \leq_{\nabla_M} \Omega$, i.e. $\Omega \sim_{\nabla_M} \omega$, contradicting (+).

Similarly from $\vdash^{\nabla_M} \Delta : (\omega \to \omega) \to \omega \to \omega$, by Lemma 17.3.45(i), we get $\omega \to \omega \leq_{\nabla_M} (\omega \to \omega) \to (\omega \to \omega)$, which implies $\omega \to \omega \leq_{\nabla_M} \omega \leq_{\nabla_M} \Omega\to\omega$, by β-soundness and (ω_{Scott}). Therefore $\Omega \sim_{\nabla_M} \omega$, contradicting (+). ∎

We finally are able to prove the main theorem.

17.3.53. THEOREM. *Let* $M \in \Lambda^{\emptyset}$. *Then* $\mathcal{F}^{\nabla_M}$ *is a non-trivial extensional* λ*-model such that* $\mathcal{F}^{\nabla_M} \models M = \Delta\Delta$.

PROOF. The model is non-trivial since clearly $\vdash^{\nabla_M} \mathsf{I} : (\omega \to \omega) \to \omega \to \omega$ and by Lemma 17.3.52 $\not\vdash^{\nabla_M} \Delta : (\omega \to \omega) \to \omega \to \omega$, therefore $\mathcal{F}^{\nabla_M} \not\models \mathsf{I} = \Delta$.

We must show that $[\![M]\!]^{\nabla_M} = [\![\Delta\Delta]\!]^{\nabla_M}$. Suppose that $W \in [\![M]\!]^{\nabla_M}$. Then

$$\begin{array}{llll} \vdash^{\nabla_M} M : W & \Rightarrow & \vdash^{\nabla_n} M : W, & \text{for some } n, \\ & \Rightarrow & \xi_i \sim_{\nabla_{n+1}} (\xi_i \to W), & \text{for some } i, \\ & \Rightarrow & \vdash^{\nabla_M} \Delta\Delta : W, & \text{by Lemma 17.3.45(iii)}, \\ & \Rightarrow & W \in [\![\Delta\Delta]\!]^{\nabla_M}. & \end{array}$$

This proves $[\![M]\!]^{\nabla_M} \subseteq [\![\Delta\Delta]\!]^{\nabla_M}$.

Now suppose $B \in [\![\Delta\Delta]\!]$, i.e. $\vdash^{\nabla_M} \Delta\Delta : B$. Then by Lemma 17.3.48 there exists A such that $\#(A) = 0$ and $\vdash^{\nabla_M} \Delta : A \leq_{\nabla_M} A \to B$. Let $A \equiv \bigcap_{i \in I} \psi_i$, with $\psi_i \in \mathbb{C} = \{\Omega, \omega, \xi_0, \ldots\}$. By Lemma 17.3.52 $\psi_i \neq \omega$. Hence it follows that $A = \Omega$ or $A \sim_{\nabla_M} \bigcap_{j \in J}(\xi_j)$, for some finite $J \subseteq \mathbb{N}$. Since $\Omega \sim_{\nabla_M} (\Omega \to \Omega)$ and $\xi_j \sim_{\nabla_M} (\xi_j \to W)$ we get $A \sim_{\nabla_M} (\Omega \to \Omega)$ or $A \sim_{\nabla_M} \bigcap_{j \in J}(\xi_j \to W_j)$. Since $A \leq_{\nabla_M} A \to B$ it follows by β-soundness that in the first case $\Omega \leq_{\nabla_M} B$ or in the second case $\bigcap_{j \in L} W_j \leq_{\nabla_M} B$, for some $L \subseteq J$. Since each W_j is in $[\![M]\!]^{\nabla_M}$, we have in both cases $B \in [\![M]\!]^{\nabla_M}$. This shows $[\![\Delta\Delta]\!] \subseteq [\![M]\!]$ and we are done. ∎

17.4. Exercises 26.2.2004:1018

17.4.1. Following the proof of Lemma 17.1.12 build a normal form X such that $x(\lambda y.a(y(\lambda z.b(yz)))[x := X]$ does not have a normal form.

17.4.2. Show that in the system induced by the type theory $\Sigma(\mathbb{C}^{\mathrm{CDZ}}, \mathrm{Scott} \cup \{(\omega\varphi), (\Omega \to \varphi)\})$, where

$$(\Omega \to \varphi) \quad \Omega \to \psi \sim \psi,$$

the terms typable with type φ in the context all whose predicates are ω are precisely the head normalizing ones.

17.4.3. Show that in the system induced by the type theory ∇^{AO}, the terms typable with type $\Omega \to \Omega$ for a suitable context are precisely the lazy normalizing ones, i.e. the terms which reduce either to an abstraction or to a (λ-free) head normal form.

17.4.4. Show that in the system induced by the type theory ∇^{EHR} the terms typable with type ν in the context all whose predicates are ν are precisely the terms which reduce either to an abstraction or to a variable using the call-by-value β-reduction rule.

17.4.5. Show that in the system induced by the type theory ∇^{Park} the terms typable with type ω in the empty context are precisely the closed terms.

17.4.6. Using the Approximation Theorem shows that

- there is no type deducible for $\Delta\Delta$ in the system $\lambda\cap_{\mathcal{B}}^{\mathrm{CDV}}$;
- $\vdash_{\Omega}^{\mathrm{BCD}} \Delta\Delta : A$ iff $A \sim_{\mathrm{BCD}} \Omega$;
- $\vdash_{\Omega}^{\mathrm{Park}} \Delta\Delta : A$ iff $\omega \leq_{\mathrm{Park}} A$.

17.4.7. Using the Approximation Theorem shows that in the system $\lambda\cap_{\Omega}^{\mathrm{AO}}$ the set of types deducible for $\Delta\Delta$ is stricty included in the set of types deducible for $\mathsf{K}(\Delta\Delta)$.

17.4.8. Using the Approximation Theorem shows that in the system $\lambda\cap_{\Omega}^{\mathrm{Scott}}$ the set of types deducible for J and I coincide.

17.4.9. Show Theorem 17.3.34 using the mapping $\mathsf{m} : \mathsf{P}(\omega)\to\mathbb{T}^{\nabla(\omega)}$ defined as

$$\begin{array}{lcl} \mathsf{m}(\emptyset) & = & \Omega, \\ \mathsf{m}(\{i\}) & = & i, \\ \mathsf{m}(u \cup \{i\}) & = & \mathsf{m}(u) \cap i. \end{array}$$

where $u\in\mathsf{P}_{\mathit{fin}}(\omega), i\in\omega$.

17.4.10. Show Theorem 17.3.37 using the mapping $\mathsf{m} : \mathsf{P}_{\mathit{fin}}(\mathsf{Pm})\to\Pi^{\mathsf{Pm}}$ defined as

$$\begin{array}{lcl} \mathsf{m}(\emptyset) & = & \Omega, \\ \mathsf{m}(\{\varphi\}) & = & \varphi, \\ \mathsf{m}(u \cup \{a\}) & = & \mathsf{m}(u) \cap \mathsf{m}(\{a\}), \\ \mathsf{m}(\{\langle u, v\rangle\}) & = & \mathsf{m}(u)\to\mathsf{m}(v). \end{array}$$

where $u, v\in\mathsf{P}_{\mathit{fin}}(\mathsf{Pm})$, $a\in\mathsf{Pm}$.

17.4.11. Show Theorem 17.3.40 using the mapping $\mathsf{m} : \mathsf{P}_{\mathit{fin}}(\mathsf{Em})\to\mathbb{T}^{\mathsf{Em}}$ defined as

$$\begin{array}{lcl} \mathsf{m}(\emptyset) & = & \Omega, \\ \mathsf{m}(\{\mathsf{a}\}) & = & \mathsf{a}, \\ \mathsf{m}(u \cup \{e\}) & = & \mathsf{m}(u) \cap \mathsf{m}(\{e\}) \\ \mathsf{m}(\{\langle u, e\rangle\}) & = & \mathsf{m}(u)\to\mathsf{m}(\{e\}). \end{array}$$

where $u\in\mathsf{P}_{\mathit{fin}}(\mathsf{Em}), e\in\mathsf{Em}, \mathsf{a}\in\mathbb{C}_{\infty}$.

Chapter 18

Further Readings

Citare:
-lavori di Kfoury, Wells, che usano unione,intersezione per i compilatori
- libro di Hankin
- lavoro di Hindley (intersection types: an introduction)

Intersection type assignment systems have been developed in a number of papers which have been cited in the previous sections. An extensive study of the syntactical properties of the several systems for intersection types is **?** (see also **?**, **?**). In particular, the papers **?** define principal type schemes for various intersection type disciplines. An exposition of the proof-theoretic properties of intersection type systems used as tools for the syntactical study of type-free λ-calculus can be found in **?**, which makes extensive use of the notion of saturated sets, called type stable sets in Section 17.1.

The paper **?** gave rise to a number of studies relating intersection type theories to domain theory and λ-models (see **?**). Many of these results are given in Section 17.3.

In his PhD thesis **?** Alessi gives an account of filter models as representations of domains. Moreover he studies certain pathologies of domains with respect to their representations, by defining a more basic category of models, which admit a good description in terms of intersection types. See also **?**. In the same thesis, and in **?**, other categories of domains, like Girard's qualitative and quantitative domains, are studied using the intersection types.

All the above-mentioned models are models of the classical λ-calculus, but intersection-type disciplines are also suitable for describing models of the $\lambda\mathbf{I}$-calculus **?**, of the lazy λ-calculus **?**, of the call-by-value λ-calculus (introduced by Plotkin in **?**) **?**, and of some extensions of the λ-calculus which include parallel features **?**.

The finitary descriptions of domains have been a major point of interest. Scott pioneered this topic with his Information Systems in **?**. Abramsky extended the theory to SFP domains in **?**, framing it into the paradigm of Stone duality. A strikingly short, but good introduction to this subject is Chapter 10 of **?**. We also suggest the reading of **?**: it is not exactly about filter models, but it gives a perspective to the webbed models (originated with Engeler models **?**), which are an alternative, though related, finitary description of λ-models. See also **?**.

Since, given an arbitrary λ-term, it is undecidable whether it can be typed using intersection types, it is interesting to look for decidable restrictions. For this topic see **?**.

Extensions of intersection types with union and quantified types, both universal and existential, have been studied in **?**. The completeness problem for such systems has been recently settled by Yokouchi in **?**.

Systems of intersection types à la Church are not trivial to define: they have been extensively studied in **?** and applied to type disciplines for object oriented languages in **?**.

Lastly we mention that intersection and union types have been successfully used to prove properties of programs, like strictness analysis, detection of dead code, etc. Pioneers in this field were the PhD thesis of Benton **?** and Jensen **?** and the paper **?**. The recent PhD thesis by Mossin **?** and Damiani **?** contain the last developments and the references to the large literature on the subject.

Bibliography

Abramsky, Samson [1991]. Domain theory in logical form, *Ann. Pure Appl. Logic* **51**(1-2), pp. 1–77. Second Annual IEEE Symposium on Logic in Computer Science (Ithaca, NY, 1987).

Abramsky, Samson and C.-H. Luke Ong [1993]. Full abstraction in the lazy lambda calculus, *Inform. and Comput.* **105**(2), pp. 159–267.

Alessi, F. [1991]. *Strutture di tipi, teoria dei domini e modelli del lambda calcolo*, Dissertation, Torino University.

Alessi, F., Mariangiola Dezani and Furio Honsell [2001]. Filter models and easy terms, *In ICTCS'01*, LNCS 2202, Springer, pp. 17–37.

Alessi, Fabio [1993]. The p model, Internal Report, Udine University.

Amadio, Roberto M. and Pierre-Louis Curien [1998]. *Domains and lambda-calculi*, Cambridge University Press, Cambridge.

Baeten, J. and B. Boerboom [1979]. ω can be anything it shouldn't be, *Indag.Math.* **41**, pp. 111–120.

van Bakel, Steffen [1992]. Complete restrictions of the intersection type discipline, *Theoret. Comput. Sci.* **102**(1), pp. 135–163.

Barbanera, Franco, Mariangiola Dezani-Ciancaglini and Ugo de'Liguoro [1995]. Intersection and union types: syntax and semantics, *Inform. and Comput.* **119**(2), pp. 202–230.

Barendregt, Henk and Sylvia Ghilezan [n.d.]. Lambda terms for natural deduction, sequent calculus and cut-elimination, *Journal of Functional Porgramming*. To appear.

Barendregt, Henk, Mario Coppo and Mariangiola Dezani-Ciancaglini [1983]. A filter lambda model and the completeness of type assignment, *J. Symbolic Logic* **48**(4), pp. 931–940 (1984).

Barendregt, H.P. [1984]. *The Lambda Calculus: its Syntax and Semantics*, revised edition, North-Holland, Amsterdam.

Böhm, Corrado and Mariangiola Dezani-Ciancaglini [1975]. λ-terms as total or partial functions on normal forms, pp. 96–121. Lecture Notes in Comput. Sci., Vol. 37.

Coppo, M. and M. Dezani-Ciancaglini [1980]. An extension of the basic functionality theory for the λ-calculus, *Notre Dame J. Formal Logic* **21**(4), pp. 685–693.

Coppo, M., M. Dezani-Ciancaglini and B. Venneri [1980]. Principal type schemes and λ-calculus semantics, *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, Academic Press, London, pp. 535–560.

Coppo, M., M. Dezani-Ciancaglini and B. Venneri [1981]. Functional characters of solvable terms, *Z. Math. Logik Grundlag. Math.* **27**(1), pp. 45–58.

Coppo, M., M. Dezani-Ciancaglini and M. Zacchi [1987]. Type theories, normal forms, and D_∞-lambda-models, *Inform. and Comput.* **72**(2), pp. 85–116.

Coppo, M., M. Dezani and G. Longo [1983]. Applicative information systems, *CAAP '83 (L'Aquila, 1983)*, Springer, Berlin, pp. 35–64.

Coppo, M., F. Honsell, M. Dezani-Ciancaglini and G. Longo [1984]. Extended type structures and filter lambda models, *Logic colloquium '82 (Florence, 1982)*, North-Holland, Amsterdam, pp. 241–262.

Curry, H.B. and R. Feys [1958]. *Combinatory Logic*, Studies in Logic and the Foundations of Mathematics I, North-Holland, Amsterdam.

Dezani-Ciancaglini, Mariangiola and Ines Margaria [1986]. A characterization of F-complete type assignments, *Theoret. Comput. Sci.* **45**(2), pp. 121–157.

Dezani-Ciancaglini, Mariangiola, Furio Honsell and Fabio Alessi [2000]. A complete characterization of the complete intersection-type theories, *in:* J. Rolim, A. Broder, A. Corradini, R. Gorrieri, R. Heckel, J. Hromkovic, U. Vaccaro and J. Wells (eds.), *ICALP Workshops 2000*, Proceedings in Informatics 8, Carleton-Scientific, Canada, pp. 287–302.

Di Gianantonio, Pietro and Furio Honsell [1993]. An abstract notion of application, *Typed lambda calculi and applications (Utrecht, 1993)*, Springer, Berlin, pp. 124–138.

Egidi, Lavinia, Furio Honsell and Simona Ronchi Della Rocca [1992]. Operational, denotational and logical descriptions: a case study, *Fund. Inform.* **16**(2), pp. 149–169. Mathematical foundations of computer science '91 (Kazimierz Dolny, 1991).

Engeler, Erwin [1981]. Algebras and combinators, *Algebra Universalis* **13**(3), pp. 389–392.

Ghilezan, Silvia [1996]. Strong normalization and typability with intersection types, *Notre Dame J. Formal Logic* **37**(1), pp. 44–52.

Girard, Jean-Yves, Yves G. A. Lafont and Paul Taylor [1989]. *Proofs and Types*, Cambridge Tracts in Theoretical Computer Science 7, Cambridge University Press.

Hindley, J. R. [1982]. The simple semantics for Coppo–Dezani–Sallé types, *International symposium on programming (Turin, 1982)*, Springer, Berlin, pp. 212–226.

Hindley, J.Roger [1983a]. The completeness theorem for typing λ-terms, *Theoret. Comput. Sci.* **22**, pp. 1–17.

Hindley, J.Roger [1983b]. The completeness theorem for typing λ-terms, *Theoret. Comput. Sci.* **22**, pp. 127–133.

Hindley, R. and G. Longo [1980]. Lambda-calculus models and extensionality, *Z. Math. Logik Grundlag. Math.* **26**(4), pp. 289–310.

Honsell, Furio and Marina Lenisa [1993]. Some results on the full abstraction problem for restricted lambda calculi, *Mathematical foundations of computer science 1993 (Gdańsk, 1993)*, Springer, Berlin, pp. 84–104.

Honsell, Furio and Marina Lenisa [1999]. Semantical analysis of perpetual strategies in λ-calculus, *Theoret. Comput. Sci.* **212**(1-2), pp. 183–209. Gentzen (Rome, 1996).

Honsell, Furio and Simona Ronchi Della Rocca [1992]. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus, *J. Comput. System Sci.* **45**(1), pp. 49–75.

Hyland, Martin [1975/76]. A syntactic characterization of the equality in some models for the lambda calculus, *J. London Math. Soc. (2)* **12**(3), pp. 361–370.

Jacopini, Giuseppe [1975]. A condition for identifying two elements of whatever model of combinatory logic, *λ-calculus and computer science theory (Proc. Sympos., Rome, 1975)*, Springer, Berlin, pp. 213–219. Lecture Notes in Comput. Sci., Vol. 37.

Johnstone, Peter T. [1986]. *Stone spaces*, Cambridge University Press, Cambridge. Reprint of the 1982 edition.

Krivine, J.-L. [1993]. *Lambda-calculus, types and models*, Ellis Horwood, New York. Translated from the 1990 French original by René Cori.

Krivine, Jean-Louis [1990]. *Lambda-calcul Types et modèles*, Masson, Paris. English translation Krivine [1993].

Leivant, Daniel [1986]. Typing and computational properties of lambda expressions, *Theoret. Comput. Sci.* **44**(1), pp. 51–68.

Longo, G. [1987]. On Church's formal theory of functions and functionals, *Technical Report 9*, Department of Computing Science, Pisa University.

Longo, Giuseppe [1983]. Set-theoretical models of λ-calculus: theories, expansions, isomorphisms, *Ann. Pure Appl. Logic* **24**(2), pp. 153–188.

Mitchell, John [1988]. Polymorphic type inference and containment, *Inform. and Comput.* **76**, pp. 211–249.

Mitchell, John [1996]. *Foundation for Programmimg Languages*, MIT Press.

Mitschke, Gerd [1976]. λ-kalkül, δ-konversion und axiomatische rekursionstheorie, *Technical Report Preprint 274*, Technische Hochschule, Darmstadt.

Park, D. [1976]. The Y-combinator in Scott's λ-calculus models (revised version), Theory of Computation Report 13, Department of Computer Science, University of Warick.

Plotkin, G. D. [1975]. Call-by-name, call-by-value and the λ-calculus, *Theoret. Comput. Sci.* **1**(2), pp. 125–159.

Plotkin, Gordon D. [1993]. Set-theoretical and other elementary models of the λ-calculus, *Theoret. Comput. Sci.* **121**(1-2), pp. 351–409. A collection of contributions in honour of Corrado Böhm on the occasion of his 70th birthday.

Pottinger, G. [1981]. The church-rosser theorem for the typed λ-calculus with surjective pairing, *Notre Dame J. Formal Logic* **22**(3), pp. 264–268.

Pottinger, Garrel [1980]. A type assignment for the strongly normalizable λ-terms, *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, Academic Press, London, pp. 561–577.

Pravato, A. [n.d.]. Categorical models of untyped lambda calculi: a monoidal approach. PHD thesis.

Pravato, A., S. Ronchi and L. Roversi [n.d.]. The call-by-value lambda calculus: a semantic investigation, *Mathematical structures in computer science*.

Ronchi Della Rocca, Simona [1988]. Lecture notes on semantics and types. Internal Report, Torino University.

Scott, Dana [1972]. Continuous lattices, *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*, Springer, Berlin, pp. 97–136. Lecture Notes in Math., Vol. 274.

Scott, Dana [1976]. Data types as lattices, *SIAM J. Comput.* **5**(3), pp. 522–587.

Scott, Dana S. [1980a]. Letter to Albert Meyer, CMU.

Scott, D.S. [1975]. Open problem, *Lambda Calculus and Computer Science Theory*, Lecture Notes in Computer Science 37, Springer, Berlin, p. 369.

Scott, D.S. [1980b]. Lambda calculus: Some models, some philosophy, *in:* J. Barwise, H. J. Keisler and K. Kunen (eds.), *The Kleene Symposium*, North-Holland Publishing Company, pp. 223–265.

Severi, Paula [1996]. *Normalisation in lambda calculus and its relation to type inference*, Dissertation, Eindhoven University of Technology.

Venneri, Betti [1996]. Private communication, Florence University.

Wadsworth, Christopher P. [1976]. The relation between computational and denotational properties for Scott's D_∞-models of the lambda-calculus, *SIAM J. Comput.* **5**(3), pp. 488–521. Semantics and correctness of programs.

Yokouchi, Hirofumi [1994]. F-semantics for type assignment systems, *Theoret. Comput. Sci.* **129**, pp. 39–77.

Zylberajch, Cécile [1991]. *Syntaxe et Semantique de la Facilité en Lambda-Calcul.*, Dissertation, Université de Paris VII.